# Math 11B
# Calculus with Applications
# Fall 2007
# Midterm Exam 1

# Solutions

1. (20 Points) Solve the following initial value problem.

$$\begin{cases} \dfrac{dN}{dt} = t^9 + t^{-11} \qquad \text{for} \quad t > 1 \\ N(1) = \dfrac{3}{10} \end{cases}$$

**Solution:**

$$N(t) = \frac{t^{10}}{10} + \frac{t^{-10}}{-10} + C = \frac{t^{10} - t^{-10}}{10} + C$$

$$\frac{3}{10} = N(1) = 0 + C = C$$

Therefore $N(t) = \dfrac{t^{10} - t^{-10} + 3}{10}$ ///

2. (20 Points) Evaluate the following integrals.

a. (5 Points) $\int \left(2x^{3/4} + 3x - 5\right) dx = \frac{8}{7}x^{7/4} + \frac{3}{2}x^2 - 5x + C$

b. (5 Points) $\int e^{2t}(1 - e^{-3t})dt = \int (e^{2t} - e^{-t})\, dt = \frac{1}{2}e^{2t} + e^{-t} + C$

c. (5 Points) $\int_1^2 (s^2 - \sqrt{s})ds = \left(\frac{1}{3}s^3 - \frac{2}{3}s^{3/2}\right)\Bigg|_1^2 = \left(\frac{8}{3} - \frac{2}{3}\cdot 2^{3/2}\right) - \left(\frac{1}{3} - \frac{2}{3}\right) = \frac{9 - 4\sqrt{2}}{3}$

d. (5 Points) $\int_0^{\pi/8} \sec^2(2y)dy = \frac{1}{2}\tan(2y)\Bigg|_0^{\pi/8} = \frac{1}{2}\left(\tan(\pi/4) - \tan(0)\right) = \frac{1}{2}\left(1 - 0\right) = \frac{1}{2}$

3. (20 Points)

a. (10 Points) Approximate the definite integral $\int_0^2 (1-x^2)\,dx$ by calculating the Riemann Sum obtained by partitioning the interval $[0,2]$ into 4 *equal subintervals*, and choosing $c_i$ to be the *right hand endpoint* of the $i^{\text{th}}$ subinterval for $1 \le i \le 4$.

**Solution:**

The partition is $P = \left[0, \frac{1}{2}, 1, \frac{3}{2}, 2\right]$, and the width of the $i^{\text{th}}$ subinterval is $\Delta x_i = \frac{1}{2}$ for $1 \le i \le 4$.

The subintervals are: $\left[0, \frac{1}{2}\right]$, $\left[\frac{1}{2}, 1\right]$, $\left[1, \frac{3}{2}\right]$, $\left[\frac{3}{2}, 2\right]$, and $c_1 = \frac{1}{2}$, $c_1 = 1$, $c_1 = \frac{3}{2}$, $c_1 = 2$. Thus

$$\int_0^2 (1-x^2)\,dx \approx \sum_{i=1}^{4}(1-c_i^2)\Delta x_i = \left(1-(1/2)^2\right)\cdot\frac{1}{2} + \left(1-1^2\right)\cdot\frac{1}{2} + \left(1-(3/2)^2\right)\cdot\frac{1}{2} + \left(1-2^2\right)\cdot\frac{1}{2}$$

$$= \frac{1}{2}\cdot\left(\frac{3}{4} + 0 - \frac{5}{4} - 3\right) = -\frac{7}{4} \qquad ///$$

b. (10 Points) Express the limit below as a definite integral, where $P = [x_0, x_1, \ldots, x_n]$ denotes a partition of the interval $[5,8]$ into $n$ subintervals, $\Delta x_i = x_i - x_{i-1}$, and $c_i \in [x_{i-1}, x_i]$ for $1 \le i \le n$. **Do not evaluate this integral.**

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} \left(3c_i^5 + e^{c_i} + \sqrt{1+c_i^2}\right)\Delta x_i$$

**Solution:**

$$\int_5^8 \left(3x^5 + e^x + \sqrt{1+x^2}\right)dx \qquad ///$$

4. (20 Points) Let $g(x) = \int_1^{3x^2+x} (1+te^t)\,dt$. Determine $g'(x)$.

**Solution:**

Using Leibniz's rule we have:

$$g'(x) = \left(1 + (3x^2+x)e^{3x^2+x}\right)(6x+1) \qquad ///$$

5. (20 Points) Use a definite integral to determine the length $L$ of the curve $y=\sqrt{1-x^2}$ from $x=0$ to $x=\frac{1}{2}$. (Hint: set up the integral for $L$, then have a look at the formulas on the back page of this exam.)

**Solution:**

$$y=(1-x^2)^{1/2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{1-x^2}} = \sqrt{\frac{1-x^2+x^2}{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$$L=\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x)\Big|_0^{1/2} = \sin^{-1}(1/2)-\sin^{-1}(0) = \frac{\pi}{6}-0 = \frac{\pi}{6}$$ ///

6. (20 Points) Find the area bounded by the curves $y=x^3$, $y=\frac{1}{x}$, and $y=8$. (Hint: set this up as an integral with respect to $y$.)

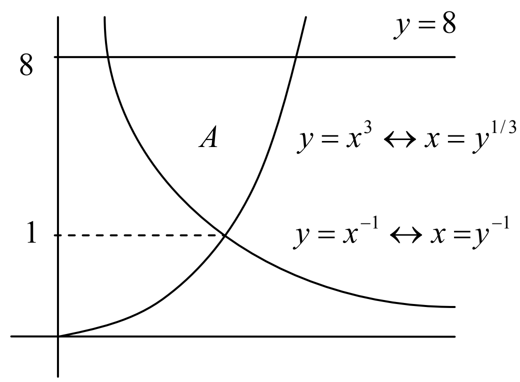
**Solution:**

With respect to $y$:

$$A=\int_1^8 (y^{1/3}-y^{-1})\,dy=\left(\frac{3}{4}y^{4/3}-\ln y\right)\Bigg|_1^8$$

$$=\left(\frac{3}{4}\cdot 8^{4/3}-\ln 8\right)-\left(\frac{3}{4}-0\right)$$

$$= 12-\ln 2^3-\frac{3}{4} = \frac{45}{4}-3\ln 2$$




With respect to $x$:

$$A=A_1+A_2=\int_{1/8}^1 (8-x^{-1})\,dx + \int_1^2 (8-x^3)\,dx$$

$$=\left(8x-\ln x\right)\Bigg|_{1/8}^1 + \left(8x-\frac{1}{4}x^4\right)\Bigg|_1^2$$

$$=(8-0)-\left(1-\ln\left(\frac{1}{8}\right)\right)+(16-4)-\left(8-\left(\frac{1}{4}\right)\right)$$

$$= 11-\ln 2^3+\frac{1}{4} = \frac{45}{4}-3\ln 2$$