MATH 11 B
HW 7 SOLUTIONS

p.424 #24 ab #26-52 even

(24) a) $\frac{1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x+2} \Rightarrow \frac{1}{(x-1)(x+2)} = \frac{(x+2)A}{(x-1)(x+2)} + \frac{(x-1)B}{(x-1)(x+2)}$

$\Rightarrow 1 = (x+2)A + (x-1)B$ Holds for all x

Let $x=-2$: $1=-3B \Rightarrow B = -\frac{1}{3}$

Let $x=1$: $1=3A \Rightarrow A = \frac{1}{3}$

b) Evaluate $\int \frac{1}{(x-1)(x+2)}dx = \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{dx}{x+2}$ (By (a))

$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C$

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(26) $\int \frac{1}{x^2+2x+5}dx$ Complete the square: $\left(-\frac{2}{2}\right)^2 = (-1)^2 = 1$

$* \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
Prove this with subst.

$= \int \frac{1}{x^2+2x+1-1+5} = \int \frac{1}{(x+1)^2+4}dx$ Let $u = x+1$, $du = dx$

$= \int \frac{1}{u^2+2^2}du \overset{(*)}{=} \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$

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(28) $\int \frac{2x-1}{(x+4)(x+1)}dx$

$= \int \frac{3}{x+4}dx - \int \frac{1}{x+1}dx$

$= 3\ln|x+4| - \ln|x+1| + C$

$\frac{2x-1}{(x+4)(x+1)} = \frac{A(x+1)}{(x+4)(x+1)} + \frac{B(x+4)}{(x+1)}$

$2x-1 = A(x+1) + B(x+4)$

$x=-1$: $-3=3B \Rightarrow B=-1$, $x=-4$: $-9=-3A \Rightarrow A=3$

(30) $\int \frac{1}{x^2+9}dx = \int \frac{1}{x^2+3^2} \overset{(*)}{=} \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

(32) $\int \frac{1}{x^2-x+2}dx$ $\left(-\frac{1}{2}\right)^2 = \frac{1}{4}$ (complete the square)

$= \int \frac{1}{x^2-x+\frac{1}{4}-\frac{1}{4}+2}dx = \int \frac{1}{(x-\frac{1}{2})^2+\frac{7}{4}}dx$ $u = x-\frac{1}{2}$, $du = dx$

$= \int \frac{1}{u^2+(\frac{\sqrt{7}}{2})^2}du \overset{(*)}{=} \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{2(x-\frac{1}{2})}{\sqrt{7}}\right) + C$

$= \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{2x-1}{\sqrt{7}}\right) + C$

(34)
$$\left.\begin{array}{r} x \\ x^2+0x+3\overline{)\,x^3+0x^2+0x+1} \\ -(x^3+0x^2+3x) \\ \hline -3x+1 \end{array}\right\} \Rightarrow \frac{x^3+1}{x^2+3} = x + \frac{-3x+1}{x^2+3}$$

$$\int \frac{x^3+1}{x^2+3}\,dx = \int x\,dx - 3\underbrace{\int \frac{x}{x^2+3}\,dx}_{\star} + \int \frac{1}{x^2+3}\,dx$$

$$(\star)\ u=x^2+3 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln(x^2+3)$$

$$\int \frac{x^3+1}{x^2+3}\,dx = \frac{1}{2}x^2 - \frac{3}{2}\ln(x^2+3) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

(36)
$$\begin{array}{r} x^2+4x+13 \\ x^2-4x+3\overline{)\,x^4+0x^3+0x^2+0x+3} \\ -(x^4-4x^3+3x^2) \\ \hline 4x-3x^2+0x \\ -(4x^3-16x^2+12x) \\ \hline 13x^2-12x+3 \\ -(13x^2-52x+39) \\ \hline 40x-36 \end{array}$$

$$\int \frac{x^4+3}{x^2-4x+3}\,dx = \int x^2+4x+13\,dx + \underbrace{\int \frac{40x-36}{x^2-4x+3}}_{\star}$$

$$(\star)\ \frac{40x-36}{(x-3)(x-1)} = \frac{A}{x-3} + \frac{B}{x-1} \Rightarrow 40x-36 = (x-1)A + (x-3)B$$

$$x=1: 4 = -2B \Rightarrow B=-2,\ x=3: 84 = 2A \Rightarrow A = 42$$

$$42\int \frac{1}{x-3}\,dx - 2\int \frac{1}{x-1}\,dx = 42\ln|x-3| - 2\ln|x-1|$$

$$\int \frac{x^4+3}{x^2-4x+3}\,dx = \frac{1}{3}x^3 + 2x^2 + 13x + 42\ln|x-3| - 2\ln|x-1| + C$$

(38)
$$\int_3^5 \frac{x}{x-1}\,dx = \int_3^5 \frac{x-1+1}{x-1}\,dx = \int_3^5 1 + \frac{1}{x-1}\,dx = x + \ln(x-1)\Big|_3^5$$
$$= 5 + \ln 4 - 3 - \ln 2 = 2 + \ln 2$$

(40)
$$\begin{array}{r} x \\ x\overline{)\,x^2+1} \\ -x^2 \\ \hline 0+1 \end{array} \qquad \int_1^2 \frac{x^2+1}{x}\,dx = \int_1^2 x + \frac{1}{x}\,dx = \int_1^2 x\,dx + \int_1^2 \frac{1}{x}\,dx$$
$$= \frac{1}{2}x^2\Big|_1^2 + \ln|x|\Big|_1^2$$
$$= \frac{3}{2} + \ln 2$$

(42)
$$\int_2^3 \frac{1}{1-x^2}\,dx \qquad \frac{1}{(1-x)(1+x)} = \frac{A}{1+x} + \frac{B}{1-x} \Rightarrow 1 = (1-x)A + (1+x)B$$

$$\frac{1}{2}\int_2^3 \frac{1}{1+x} + \frac{1}{1-x}\,dx \qquad x=1: B=\frac{1}{2},\ x=-1: A=\frac{1}{2}$$

$$\frac{1}{2}\left(\ln(1+x) - \ln|1-x|\right)\Big|_2^3 = \frac{1}{2}(\ln 4 - \ln 2 - \ln 3 + \ln 1) = \frac{1}{2}\ln\frac{2}{3}$$

<u>Notice</u>: $\int \frac{1}{(x-a)^2}dx = -(x-a)^{-1} + c$

(let $u=x-a$, $du=dx \Rightarrow \frac{1}{u^2}=u^{-2}$)

(44) $\int_0^1 x\tan^{-1}x\,dx$ | $u=\tan^{-1}x$ $dv=x$

$=\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\int\frac{x^2}{1+x^2}dx$ | $du=\frac{1}{1+x^2}$ $v=\frac{1}{2}x^2$

$\frac{x^2}{1+x^2}=\frac{x^2+1-1}{1+x^2}=1-\frac{1}{1+x^2}$

$\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\int 1-\frac{1}{1+x^2}dx=\frac{1}{2}\left(x^2\tan^{-1}x-x+\tan^{-1}x\right)\Big|_0^1=\frac{1}{2}\left(\frac{\pi}{4}-1+\frac{\pi}{4}\right)=\frac{\pi}{4}-\frac{1}{2}$

(46) $\frac{1}{x^2(x-1)^2}=\frac{a}{x}+\frac{b}{x^2}+\frac{c}{(x-1)}+\frac{d}{(x-1)^2}\Rightarrow ax(x-1)^2+b(x-1)^2+c(x^2)(x-1)+dx^2=1$

$2\int\frac{1}{x}dx+\int x^{-2}dx-2\int\frac{1}{x-1}dx+\int\frac{1}{(x-1)^2}dx$

$2\ln|x|-x^{-1}-2\ln|x-1|-(x-1)^{-1}+c$

$(a+c)x^3+(b-2a-c+d)x^2+(a-2b)x+b=1$

$\Rightarrow b=1,\ a+c=0,\ b-2a-c+d=0,\ a-2b=0$

$\Rightarrow a=2,\ c=-2,\ d=1$

(48) $\frac{2x^2+2x-1}{x^3(x-3)}=\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x^3}+\frac{d}{(x-3)}\Rightarrow ax^2(x-3)+bx(x-3)+c(x-3)+dx^3=2x^2+2x-1$

$-\frac{23}{27}\int\frac{1}{x}dx-\frac{5}{9}\int x^{-2}dx+\frac{1}{3}\int x^{-3}dx+\frac{23}{27}\int(x-3)dx$

$-\frac{23}{27}\ln|x|+\frac{5}{9}x^{-1}-\frac{1}{6}x^{-2}+\frac{23}{27}\ln|x-3|+C$

$\Rightarrow a+d=0,\ -3a+b=2,\ -3b+c=2,\ -3c=-1$

$\Rightarrow c=\frac{1}{3},\ -3b=\frac{5}{3}\Rightarrow b=\frac{-5}{9}\Rightarrow -3a=\frac{23}{9}\Rightarrow a=\frac{-23}{27}$

(50) $\frac{1}{(x^2-x-2)^2}=\frac{1}{((x-2)(x+1))^2}=\frac{1}{(x-2)^2(x+1)^2}=\frac{a}{x-2}+\frac{b}{(x-2)^2}+\frac{c}{(x+1)}+\frac{d}{(x+1)^2}$

$-\frac{2}{27}\int\frac{1}{x-2}+\frac{1}{9}\int\frac{1}{(x-2)^2}+\frac{2}{27}\int\frac{1}{x+1}$

$+\frac{1}{9}\int\frac{1}{(x+1)^2}dx$

$=-\frac{2}{27}\ln|x-2|-\frac{1}{9}(x-2)^{-1}$

$+\frac{2}{27}\ln|x+1|-\frac{1}{9}(x+1)^{-1}+C$

OR $=\frac{2}{27}\ln\left|\frac{x+1}{x-2}\right|-\frac{1}{9}\left(\frac{2x-1}{(x-2)(x+1)}\right)+C$

$\Rightarrow (x-2)(x+1)^2a+(x+1)^2b+(x+1)(x-2)^2c+(x-2)^2d=1$

$\Rightarrow (x^3-3x-2)a+(x^2+2x+1)b+(x^3-3x^2+4)c+(x^2-4x+4)d=1$

$\Rightarrow a+c=0,\ b-3c+d=0,\ -3a+2b-4d=0,\ 2a+b+4c+4d=1$

Also, $x=-1$: $9d=1\Rightarrow d=\frac{1}{9}$

$x=2$: $9b=1\Rightarrow b=\frac{1}{9}$

$\frac{1}{9}-3c+\frac{1}{9}=0\Rightarrow -3c=-\frac{2}{9}\Rightarrow c=\frac{2}{27}\Rightarrow a=-\frac{2}{27}$

(52) $\frac{1}{(x+1)^2(x^2+1)}=\frac{a}{(x+1)}+\frac{b}{(x+1)^2}+\frac{cx+d}{(x^2+1)}\leftarrow$ irreducible

$\frac{1}{2}\int\frac{1}{x+1}+\frac{1}{2}\int\frac{1}{(x+1)^2}-\frac{1}{2}\underbrace{\int\frac{x}{x^2+1}dx}_{*}$

$*u=x^2+1$ $du=2x$

$=\frac{1}{2}\ln|x+1|-\frac{1}{2}(x+1)^{-1}-\frac{1}{4}\ln|x^2+1|+C$

$\Rightarrow (x+1)(x^2+1)a+(x^2+1)b+(x+1)^2(cx+d)=1$

$\Rightarrow (x^3+x^2+x+1)a+(x^2+1)b+(x^2+2x+1)(cx+d)=1$

$\Rightarrow a+c=0,\ a+b+2c+d=0,\ a+c+2d=0,\ a+b+d=1$

Also, $x=-1$: $2b=1\Rightarrow b=\frac{1}{2}$ <u>Elimination</u>:

$a+2c+d=-\frac{1}{2}$

$+\ -2(a+c+2d=0)$

$-a-3d=-\frac{1}{2}$

$a+d=\frac{1}{2}$

$-a-3d=-\frac{1}{2}$

$-2d=0\Rightarrow d=0\Rightarrow a=\frac{1}{2}$

$c=-\frac{1}{2}$

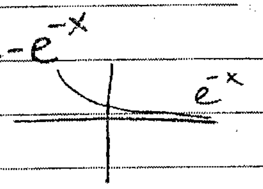
p.442 #2-26 even, 30, 36ab, 38ab

② $\lim_{z\to\infty} \int_0^z xe^{-x}dx \quad u=x \quad dv=e^{-x} \quad du=dx \quad v=-e^{-x}$

$\lim_{z\to\infty} \left(-xe^{-x}+\int e^{-x}dx\right)\Big|_0^z = (-xe^{-x}-e^{-x})\Big|_0^z$

$\lim_{z\to\infty} (-ze^{-z}-e^{-z}-1) = 0-0-1=-1$

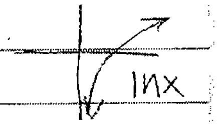
$\left(\lim_{z\to\infty} \frac{-z}{e^z} = \frac{\infty}{\infty} \quad L'H: \lim_{z\to\infty} \frac{-1}{e^z} = 0\right)$



④ $\int_e^z \frac{dx}{x(\ln x)^2} \quad u=\ln x \quad du=\frac{1}{x}dx$

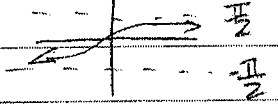
$\int u^{-2}du = -(\ln x)^{-1}\Big|_e^z = -\frac{1}{\ln z}+\frac{1}{\ln e} = -\frac{1}{\ln z}+1$

$\lim_{z\to\infty}\left(-\frac{1}{\ln z}+1\right) = 0+1 = 1$



⑥ $\int_z^{-1} \frac{1}{1+x^2}dx = \tan^{-1}x\Big|_z^{-1} = -\frac{\pi}{4}-\tan^{-1}z$

$\lim_{z\to-\infty}\left(-\frac{\pi}{4}-\tan^{-1}z\right) = -\frac{\pi}{4}-\left(-\frac{\pi}{2}\right) = -\frac{\pi}{4}+\frac{\pi}{2} = \frac{\pi}{4}$



⑧ $\int_0^\infty \overset{①}{xe^{-x^2/2}}dx + \int_{-\infty}^0 \overset{②}{xe^{-x^2/2}}dx$

① $\lim_{z\to\infty} \int_0^z xe^{-x^2/2}dx \quad u=\frac{-x^2}{2} \quad du=-xdx$

$\lim_{z\to\infty} -\int e^u du = \lim_{z\to\infty}\left(-e^{-x^2/2}\Big|_0^z\right) = \lim_{z\to\infty}\left(-e^{-z^2/2}+1\right) = 1$

② $\lim_{z\to\infty} \int_{-z}^0 xe^{-x^2/2}dx = \lim_{z\to\infty}\left(-e^{-x^2/2}\Big|_{-z}^0\right) = \lim_{z\to\infty}\left(-1+e^{-z^2/2}\right) = -1$

$1-1=0$

⑩ $\int_0^\infty \overset{①}{x^3e^{-x^4}}dx + \int_{-\infty}^0 \overset{②}{x^3e^{-x^4}}dx, \; u=-x^4 \quad du=4x^3 \Rightarrow -\frac{1}{4}du = x^3dx$

① $-\frac{1}{4}\int e^u du = -\frac{1}{4}e^{-x^4}\Big|_0^z = -\frac{1}{4}e^{-z^4}+\frac{1}{4}$

$\lim_{z\to\infty}\left(-\frac{1}{4}e^{-z^4}+\frac{1}{4}\right) = -\frac{1}{4}\cdot 0+\frac{1}{4} = \frac{1}{4}$

② $-\frac{1}{4}e^{-x^4}\Big|_{-z}^0 = -\frac{1}{4}+\frac{1}{4}e^{-z^4} \quad \lim_{z\to\infty}\left(-\frac{1}{4}+\frac{1}{4}e^{-z^4}\right) = -\frac{1}{4}$

$\frac{1}{4}-\frac{1}{4} = 0$

⑫ $\int_1^e \frac{dx}{x\sqrt{\ln x}} = \lim_{c\to1^+} \int_c^e \frac{dx}{x\sqrt{\ln x}}$ (ln(1)=0, so undefined @ x=1)

$\int u^{-1/2}du = 2(\ln x)^{1/2}\Big|_c^e \quad u=\ln x \quad du=\frac{1}{x}dx$

$= 2(1-\sqrt{\ln c}) \quad \lim_{c\to1^+} 2(1-\sqrt{\ln c}) = 2(1-\sqrt{0}) = 2$

⑭ $\int_{-2}^0 \frac{dx}{(x+1)^{1/3}} = \lim_{c\to-1^-} \overset{①}{\int_{-2}^c \frac{dx}{(x+1)^{1/3}}} + \lim_{c\to-1^+} \overset{②}{\int_c^0 \frac{dx}{(x+1)^{1/3}}}$ (undefined @ x=-1)

① $\int_{-2}^c (x+1)^{-1/3}dx = \frac{3}{2}(x+1)^{2/3}\Big|_{-2}^c = \frac{3}{2}\left[(c+1)^{2/3}-1\right] \longrightarrow -\frac{3}{2}$ as $c\to-1$

② $\frac{3}{2}(x+1)^{2/3}\Big|_c^0 = \frac{3}{2}\left(1-(c+1)^{2/3}\right) \to \frac{3}{2}$ as $c\to-1$

$-\frac{3}{2}+\frac{3}{2} = 0$

(16) $\lim_{c\to1^-}\overset{①}{\int_0^c\frac{dx}{(x-1)^{2/5}}} + \lim_{c\to1^+}\overset{②}{\int_c^2\frac{dx}{(x-1)^{2/5}}}$ (undefined @ $x=1$)

① $\int(x-1)^{-2/5}dx = \frac{5}{3}(x-1)^{3/5}\Big|_0^c = \frac{5}{3}[(c-1)^{3/5}+1] \longrightarrow \frac{5}{3}$ as $c\to1$

② $\frac{5}{3}(x-1)^{3/5}\Big|_c^2 = \frac{5}{3}[1-(c-1)^{3/5}] \to \frac{5}{3}$ as $c\to1$

$\frac{5}{3}+\frac{5}{3}=\frac{10}{3}$

(18) $\lim_{c\to\infty}\int_1^c x^{-1/3}dx = \lim_{c\to\infty}\left[\frac{3}{2}x^{2/3}\Big|_1^c\right] = \lim_{c\to\infty}\left[\frac{3}{2}c^{2/3}-\frac{3}{2}\right]=\infty$ divergent

(20) $\lim_{c\to0^+}\int_c^4 x^{-1/4}dx = \lim_{c\to0^+}\left[\frac{4}{3}x^{3/4}\Big|_c^4\right] = \frac{4}{3}(4)^{3/4}-\frac{4}{3}(0)=\frac{4}{3}(2\sqrt{2})=\frac{8\sqrt{2}}{3}$

(22) $\lim_{c\to1^-}\overset{①}{\int_0^c(x-1)^{-4}} + \lim_{c\to1^+}\overset{②}{\int_c^2(x-1)^{-4}}$ (undefined @ $x=1$)

① $-\frac{1}{3}(x-1)^{-3}\Big|_0^c = -\frac{1}{3}[(c-1)^{-3}+1]$ & $\lim_{c\to1^-}\frac{1}{c-1}=-\infty$

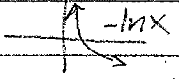
so divergent overall

(24) $\lim_{c\to-1^-}\int_c^0\frac{1}{\sqrt{x+1}}dx$ (since undefined @ $x=-1$)

$\int_c^0(x+1)^{-1/2}dx = 2(x+1)^{1/2}\Big|_c^0 = 2(1-\sqrt{c+1}) \to 2$ as $c\to-1$

so $\int_{-1}^0\frac{1}{\sqrt{x+1}}dx=2$

(26) $\lim_{c\to1^+}\int_c^e\frac{dx}{x\ln x}$ (since undefined @ $x=1$, $\ln(1)=0$)



$u=\ln x$ $du=\frac{1}{x}$ $\int u^{-1}du = \ln|\ln x|\Big|_c^e = 0-\ln|\ln c| \to \infty$ as $c\to0$

so divergent

(30) $\int_{-\infty}^{\infty}x\cdot xe^{-x^2/2}dx$ $u=x$ $dv=xe^{-x^2/2}$

$du=dx$ $v=-e^{-x^2/2}$

$-xe^{-x^2/2}\Big|_{-\infty}^{\infty}+\int_{-\infty}^{\infty}e^{-x^2/2}dx$ $-xe^{-x^2/2}\Big|_0^c = -ce^{-c^2/2}\xrightarrow{c\to\infty}0$

$-xe^{-x^2/2}\Big|_{-c}^0 = 0+ce^{-c^2/2}\xrightarrow{c\to\infty}0$

So $\int_{-\infty}^{\infty}x^2e^{-x^2/2}dx = 0+\sqrt{2\pi}=\sqrt{2\pi}$

(36) a) $x^4\le1+x^4 \Rightarrow x^2\le\sqrt{1+x^4} \Rightarrow \frac{1}{\sqrt{1+x^4}}\le\frac{1}{x^2}$ & $\sqrt{1+x^4}\ge0 \Rightarrow 0\le\frac{1}{\sqrt{1+x^4}}$

b) $\int_1^{\infty}\frac{1}{x^2}dx = -x^{-1}\Big|_1^{\infty} = -\frac{1}{x}\Big|_1^{\infty} = 0+1=1$

So $0\le\int_1^{\infty}\frac{1}{\sqrt{1+x^4}}\le\int_1^{\infty}\frac{1}{x^2}=1 \Rightarrow \int_1^{\infty}\frac{1}{\sqrt{1+x^4}}$ is convergent

(38) a) For $x\ge1$, $x\ge\ln x \Rightarrow x+\ln x\le2x \Rightarrow \sqrt{x+\ln x}\le\sqrt{2x} \Rightarrow \frac{1}{\sqrt{2x}}\le\frac{1}{\sqrt{x+\ln x}}$

Also $\sqrt{2x}\ge0 \Rightarrow \frac{1}{\sqrt{2x}}>0$. So $0<\frac{1}{\sqrt{2x}}\le\frac{1}{\sqrt{x+\ln x}}$

b) $\lim_{c\to\infty}\int_1^c\frac{1}{\sqrt{2x}}dx$ $\frac{1}{\sqrt{2}}\int_1^c\frac{1}{\sqrt{x}}dx = \frac{2}{\sqrt{2}}x^{1/2}\Big|_1^c = \frac{2}{\sqrt{2}}(\sqrt{c}-1)\xrightarrow{c\to\infty}\infty$

Since $\int_1^{\infty}\frac{1}{\sqrt{2x}}\le\int_1^{\infty}\frac{1}{\sqrt{x+\ln x}}$ & $\int_1^{\infty}\frac{1}{\sqrt{2x}}dx$ diverges, $\int_1^{\infty}\frac{1}{\sqrt{x+\ln x}}$ diverges