Math 11b Review

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1 Review

This compilation of definitions and theorems is meant to be a useful reference for Math 11b. It should also be useful for students who will eventually take Math 22. This is by no means a complete "compilation", but should give some useful things to think about. All of this material was taken from *Calculus for Biology and Medicine* by Claudia Neuhauser, and should be read over in the book[1] (look carefully over examples). Also, these definitions and theorems can often be visualized with graphs and pictures, seeing these pictures is a great way to understand the definitions, so look in the book at the examples!

2 Review of Derivates

Here are some basic definitions and properties of derivatives, these are neccessary when checking to see if you have the correct antiderivatieve, and some of thier properties are the same as integrals.

2.1 Properties of Derivatives

Property 1 (Scalar Multiplication)

$$\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx}(f(x)) \qquad c \in \Re \tag{1}$$

Property 2 ("Derivative of a sum is the sum of the derivatives")

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] \tag{2}$$

Property 3 (Product Rule)

$$\frac{d}{dx}[f(x) \cdot g(x)] = \frac{d}{dx}[f(x)] \cdot g(x) + f(x) \cdot \frac{d}{dx}[f(x)] \tag{3}$$

[1] There are *many* books in the library that cover the same material, sometimes it is good to look over another book to get someone else's "take" on the same material ...

Property 4 (Chain Rule)

$$\frac{d}{dx} f[g(x)] = f'[g(x)] \cdot g'(x) \tag{4}$$

3 Review of Integrals

3.1 Definite Integrals

Definition 1 (Definite Integral) *Let $P = [x_0, x_1, x_2, \ldots, x_n]$ be a partition of $[a, b]$ and set $\Delta x_k = x_k - x_{k-1}$ and $c_k \in [x_{k-1}, x_k]$. The **definite integral** of f from a to b is*

$$\int_a^b f(x)\, dx = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k) \Delta x_k \tag{5}$$

*if the limit exists, in which case f is said to be (Riemann) **integrable** on the interval $[a, b]$.*

3.2 Properties of Integrals

Assume that f and g are integrable on $[a, b]$.

Property 1

$$\int_a^a f(x)\, dx = 0\ . \tag{6}$$

Property 2

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx\ . \tag{7}$$

Property 3 (Scalar Multiplication)

$$\int_a^b k \cdot f(x)\, dx = k \cdot \int_a^b f(x)\, dx\ , \qquad k \in \Re\ . \tag{8}$$

Property 4 ("Integral of the sum is the sum of the integrals")

$$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx\ . \tag{9}$$

Property 5 *If f is integrable over an interval containing the three numbers a, b, and c, then*

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx\ . \tag{10}$$

Property 6 *If $f(x) \geq 0$ on $[a, b]$, then*

$$\int_a^b f(x)\, dx \geq 0\ . \tag{11}$$

Property 7 *If* $f(x) \leq g(x)$ *on [a,b], then*

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx\ . \tag{12}$$

Property 8 *If* $m \leq f(x) \leq M$ *on* $[a,b]$*, then*

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)\ . \tag{13}$$

3.3 The Fundamental Theorem of Calculus

Fundamental Theorem of Calculus 1 (Part I) *If* f *is continuous on* $[a,b]$*, then the function* F *defined by*

$$F(x) = \int_a^x f(u)\,du\ , \qquad a \leq x \leq b \tag{14}$$

is continuous on (a,b)*,with*

$$\frac{d}{dx}F(x) = f(x) \tag{15}$$

Theorem 1 (Leibniz's Rule) *If* $g(x)$ *and* $h(x)$ *are differentiable functions and* $f(u)$ *is continuous for* u *between* $g(x)$ *and* $h(x)$*, then*

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(u)\,du = f[h(x)]h'(x) - f[g(x)]g'(x) \tag{16}$$

Fundamental Theorem of Calculus 2 (Part II) *Assume that* f *is continuous on* $[a,b]$*; then*

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{17}$$

where $F(x)$ *is an antiderivative of* $f(x)$*, that is,* $F'(x) = f(x)$*.*

Theorem 2 (Area) *If* f *and* g *are continuous on* $[a,b]$*, with* $f(x) \geq g(x)$ *for all* $x \in [a,b]$*, then the area of the region between the curves* $y = f(x)$ *and* $y = g(x)$ *from* a *to* b *is equal to*

$$Area = \int_a^b [f(x) - g(x)]\,dx\ . \tag{18}$$

3.4 Applications of Integration

Theorem 3 (Average Value) *Assume that* $f(x)$ *is a continuous function on* $[a,b]$*. The average value of* f *on the interval* $[a,b]$ *is*

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx\ . \tag{19}$$

4 Integration Techniques

Here are some of the most widely used integration techniques[2].

4.1 Methods of Integration

Rule 1 (Substitution Rule for Indefinite Integrals) *If $u = g(x)$, then*

$$\int f[g(x)]g'(x)dx = \int f(u)du \ . \tag{20}$$

Rule 2 (Substitution Rule for Definite Integrals) *If $u = g(x)$, then*

$$\int_a^b f[g(x)]g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)du \ . \tag{21}$$

Rule 3 (Integration by Parts Rule) *If $u(x)$ and $g(x)$ are differentiable functions, then*

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx \tag{22}$$

or, in short form

$$\int udv = uv - \int vdu \ . \tag{23}$$

Method 1 (Partial Fraction Decomposition) *For integrands of the form*

$$\frac{P(x)}{Q(x)} \ , \tag{24}$$

where $P(x)$ and $Q(x)$ are polynomials of degree n and m respectivly, one might use PFD. If $n \geq m$, then use long division. If $n \leq m$, then use partial fraction decomposition. There are four possibilities for the denominator Q(x):

1. *Linear Factors*

$$\frac{P(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d} \tag{25}$$

2. *Repeated Linear Factors*

$$\frac{P(x)}{(ax+b)(cx+d)^3} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2} + \frac{D}{(cx+d)^3} \tag{26}$$

3. *Irreducible Quadratic Factors*

$$\frac{P(x)}{(ax^2+b)(cx^2+d)} = \frac{Ax+B}{ax^2+b} + \frac{Cx+D}{cx^2+d} \tag{27}$$

[2]Remember that one should always try to simplify the integrand *algebraicly* first, then use the following methods (i.e. try something easy and then something harder).

4. Repeated Irreducible Quadratic Factors

$$\frac{P(x)}{(ax^2+b)^2(cx^2+d)(ex+f)} = \frac{Ax+B}{ax^2+b} + \frac{Cx+D}{(ax^2+b)^2} + \frac{Ex+F}{cx^2+d} + \frac{G}{ex+f} \tag{28}$$

Next find the coefficients (i.e. A, B, C, ...) by the method of comparing coefficients.

4.2 Improper Integrals

4.2.1 Type 1: Unbounded Intervals

Theorem 4 (Unbounded Intervals)

$$\int_a^\infty f(x)\,dx = \lim_{z\to\infty} \int_a^z f(x)\,dx \tag{29}$$

$$\int_{-\infty}^a f(x)\,dx = \lim z \to -\infty \int_z^a f(x)\,dx \tag{30}$$

Theorem 5 *Let $f(x)$ be continuous on the interval $[a, \infty)$. If*

$$\lim_{z\to\infty} \int_a^z f(x)\,dx \tag{31}$$

exists and has a finite value, we say that the improper integral

$$\int_a^\infty f(x)\,dx \tag{32}$$

***converges**, and define*

$$\int_a^\infty f(x)\,dx = \lim_{z\to\infty} \int_a^z f(x)\,dx\ , \tag{33}$$

*otherwise, we say that the improper integral **diverges**.*

Theorem 6

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx\ ^3 \tag{34}$$

[3] This can also be understood fromthe properties of Integrals

4.2.2 Type 2: Discontinuous Integrand

Integrals with an integrand that has a discontinuity in the interval of integration have discontinuous integrands.

Theorem 7

$$\int_a^b f(x)\, dx = \lim_{c \to a^+} \int_c^a f(x)\, dx \tag{35}$$

Theorem 8

$$\int_a^b f(x)\, dx = \lim_{c \to b^-} \int_a^c f(x)\, dx \tag{36}$$

4.2.3 Comparison Rule for Improper Integrals

Sometimes the convergence or divergence of an integral cannot be found simply by taking the limit because the integrand is complicated. One method to deal with this is to compare the integrand (and hence the integral) to an integrand that is known to converge or diverge. This is known as the comparison rule.

Theorem 9 (Comparison Rule for Convergence) *We assume that $f(x) \geq 0$ for $x \geq a$. Suppose we wish to show that $\int_a^\infty f(x)\, dx$ is convergent. It is enough to find a function $g(x)$ such that $g(x) \geq f(x)$ for all $x \geq a$ and $\int_a^\infty g(x)\, dx$ is convergent.*

$$0 \leq \int_a^\infty f(x)\, dx \leq \int_a^\infty g(x)\, dx\ . \tag{37}$$

If $\int_a^\infty g(x)\, dx < \infty$, it follows that $\int_a^\infty f(x)\, dx$ is convergent.

Theorem 10 (Comparison Rule for Divergence) *We again assume that $f(x) \geq 0$ for all $x \geq a$. Suppose we now wish to show that $\int_a^\infty f(x)\, dx$ is divergent. It is then enough to find a function $g(x)$ such that $0 \leq g(x) \leq f(x)$ for all $x \geq a$ and $\int_a^\infty g(x)\, dx$ is divergent.*

$$\int_a^\infty f(x)\, dx \geq \int_a^\infty g(x)\, dx \geq 0\ . \tag{38}$$

If $\int_a^\infty g(x)\, dx$ is divergent, it follows that $\int_a^\infty f(x)\, dx$ is divergent.

4.3 Tables of [Important] Integrals

1.

$$\int x^n\, dx = \frac{1}{n+1} x^{n+1} + c\ , n \neq -1 \tag{39}$$

2.

$$\int \frac{1}{x}\, dx = ln|x| + c \tag{40}$$

3.

$$\int e^x \, dx = e^x + c \tag{41}$$

4.

$$\int sinx \, dx = -cosx + c \tag{42}$$

5.

$$\int \frac{1}{1+x^2} \, dx = arctanx + c \tag{43}$$

4.4 Taylor and MacLaurin Expansions

Taylor and MacLaurin series expansions are used to describe the behavior of functions (locally) around a certain point. The error in the approximation grows as you move farther from the point of expansion, adding more terms gives a more precise approximation for values 'far' from the point of expansion.

Definition 2 (MacLaurin Series) *The Maclaurin Series Expansion of $f(x)$ about $x = 0$ is defined as*

$$f(x) \equiv \sum_{n=0}^{\infty} \frac{f^(n)(0)}{n!} x^n \; . \tag{44}$$

The first few terms look like this:

$$f(x) \approx f(0) + \frac{f'(0)}{1!} x^1 + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots \tag{45}$$

Definition 3 *The Taylor Series Expansion of $f(x)$ about $x = a$ is defined as*

$$f(x) \equiv \sum_{n=0}^{\infty} \frac{f^(n)(a)}{n!} (x-a)^n \; . \tag{46}$$

The first few terms look like this [4]:

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \tag{47}$$

Theorem 11 *An expansion of a function $f(x)$ can be written in terms of its n^{th} degree Taylor Polynomial plus a remainder term:*

$$f(x) = T_n(x) + R_{n+1}(x) \; . \tag{48}$$

Theorem 12 (Remainder Term) *There exists a c between a and x such that the error term in Taylor's Formula (from above) is of the form*

$$R_{n+1}(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1} \tag{49}$$

[4] By setting $a = 0$ in the Taylor Series definition you recover the MacLaurin Series definition.

5 Differential Equations

5.1 Solving Differential Equations

Method 2 (Solving First Order Seperable Differential Equations)

1. *Separate variables,*
2. *Integrate both sides,*
3. *Solve for the dependent variable (i.e. if you are given* $\frac{dy}{dx} = y$*, solve for* $y(x)$*),*
4. *Plug in initial value and solve for the integration constant* c*,*
5. *Check your solution to see if it satisfies the differential equation .*

5.2 Equilibria and Their Stability

When discussing equilibria, we are dealing with a seperable first order differential equation of the form:

$$\frac{dy}{dx} = g(y)\ , \tag{50}$$

which is known as a *autonomous differential equation*.

Theorem 13 *If* $\hat{y}$ *satisfies*

$$g(\hat{y}) = 0 \tag{51}$$

then $\hat{y}$ *is an* ***equilibrium*** *of*

$$\frac{dy}{dx} = g(y) \tag{52}$$

Theorem 14 (Stability Criterion) *Consider the differential equation*

$$\frac{dy}{dx} = g(y) \tag{53}$$

where $g(y)$ *is a differentiable function. Assume that* $\hat{y}$ *is an equilibrium, that is,* $g(\hat{y}) = 0$*. Then*

1. $\hat{y}$ *is locally stable if* $g'(\hat{y}) < 0$ *,*
2. $\hat{y}$ *is unstable if* $g'(\hat{y}) > 0$ *,*
3. $\hat{y}$ *is semi-stable if* $g'(\hat{y}) = 0$ *.*