## 8.2 EQUILIBRIA AND STABILITY

WE RETURN TO THE STUDY OF AUTONOMOUS EQUATIONS

(1) $$\frac{dy}{dx} = g(y)$$

SUPPOSE $y_1 \in \mathbb{R}$ IS A ROOT OF THE FUNCTION $g$, i.e. SUPPOSE

(2) $$g(y_1) = 0$$

THEN THE CONSTANT FUNCTION

(3) $$y(x) = y_1$$

IS A SOLUTION TO (1).

CHECK:

$$\text{LHS} = \frac{dy}{dx} = y'(x) = 0$$

$$\text{RHS} = g(y(x)) = g(y_1) = 0 \quad (\text{BY } (2)).$$

THUS LHS = RHS FOR ALL $x$. THUS (3) IS A SOLUTION TO (1)

ON THE OTHER HAND, SUPPOSE SOME CONSTANT FUNCTION

$$y(x) = c$$

IS A SOLUTION TO (1). THEN

$$g(c) = g(y(x)) = \frac{dy}{dx} = 0 .$$

THUS A CONSTANT FUNCTION SOLVES (1) IF AND ONLY IF THAT CONSTANT IS A ROOT OF THE FUNCTION $g$.

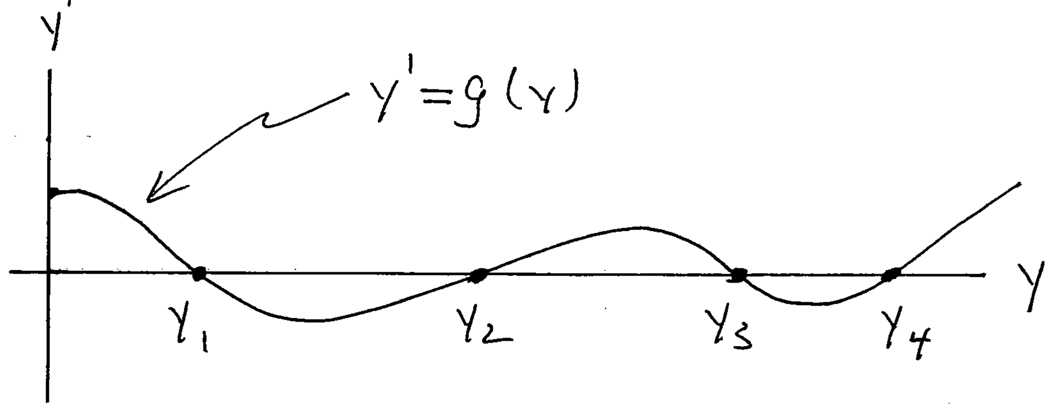
SUCH SOLUTIONS ARE CALLED <u>EQUILIBRIUM SOLUTIONS</u> OR SIMPLY <u>EQUILIBRIA</u>

## 8.2.1 STABILITY

AS WE'VE SEEN, EQUATION (1) HAS IN GENERAL, INFINITELY MANY SOLUTIONS.

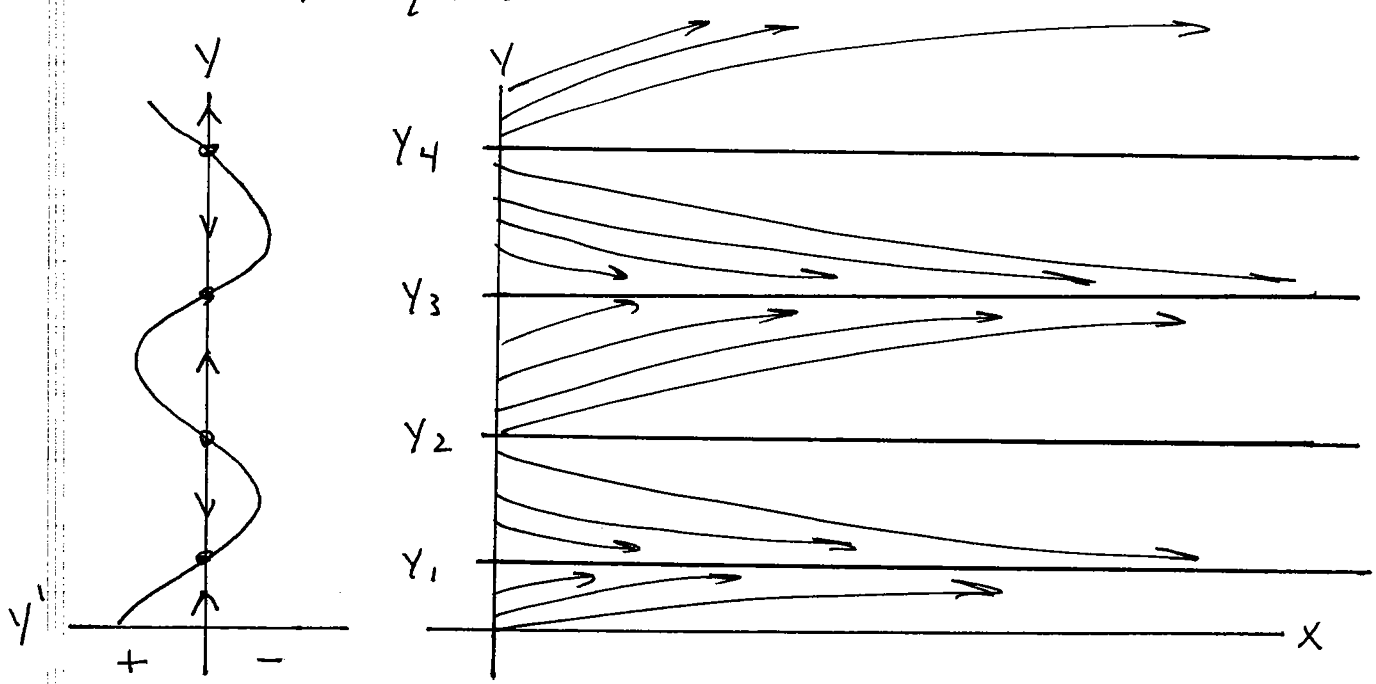
WE CAN GAIN A GOOD QUALITATIVE PICTURE OF THE ENTIRE FAMILY OF SOLUTIONS TO (1) BY STUDYING THE GRAPH OF $g(y)$!

FOR EXAMPLE




WE SEE THAT $y_1, y_2, y_3, y_4$ ARE EQUILIBRIA.

IT IS HELPFUL TO TURN THIS GRAPH ON ITS SIDE AND COMPARE IT TO THE GRAPHS IN THE X-Y PLANE OF SOLUTIONS $y(x)$.



OBSERVE THAT

$$g(y) > 0 \Rightarrow \frac{dy}{dx} > 0 \Rightarrow y(x) \text{ is INCREASING}$$

AND

$$g(y) > 0 \Rightarrow \frac{dy}{dx} < 0 \Rightarrow y(x) \text{ is DECREASING}.$$

NOTICE THAT ANY INITIAL VALUE $y_0$ WHICH IS NEAR $y_3$ FOR INSTANCE MUST PRODUCE A SOLUTION THAT APPROACHES $y = y_3$ ASYMPTOTICALLY.

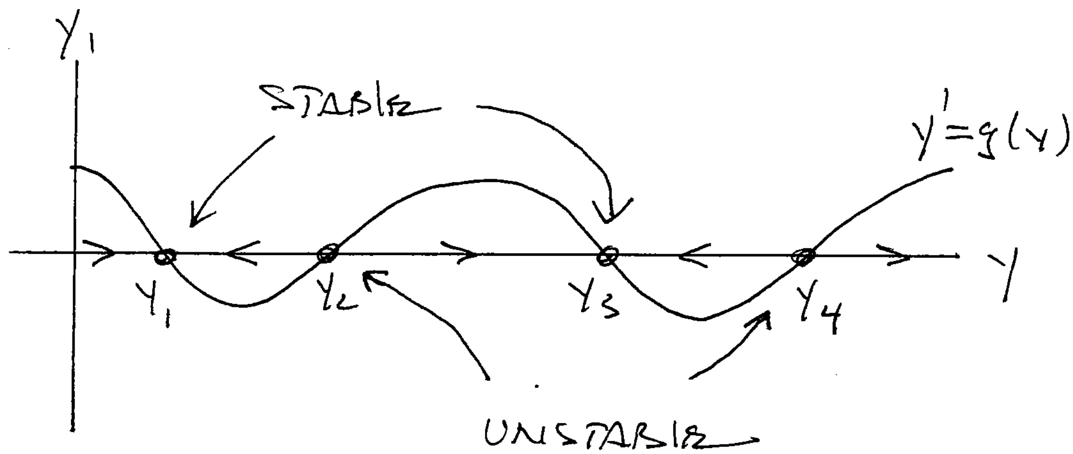
IF $y_0 \in (y_2, y_3)$ THEN $\frac{dy}{dx} > 0$, WHILE IF $y_0 \in (y_3, y_4)$ THEN $\frac{dy}{dx} < 0$.

(NOTE ALSO: NO TWO SOLUTIONS TO (1) CAN CROSS, SINCE INITIAL CONDITIONS UNIQUELY DETERMINE A SOLUTION.)

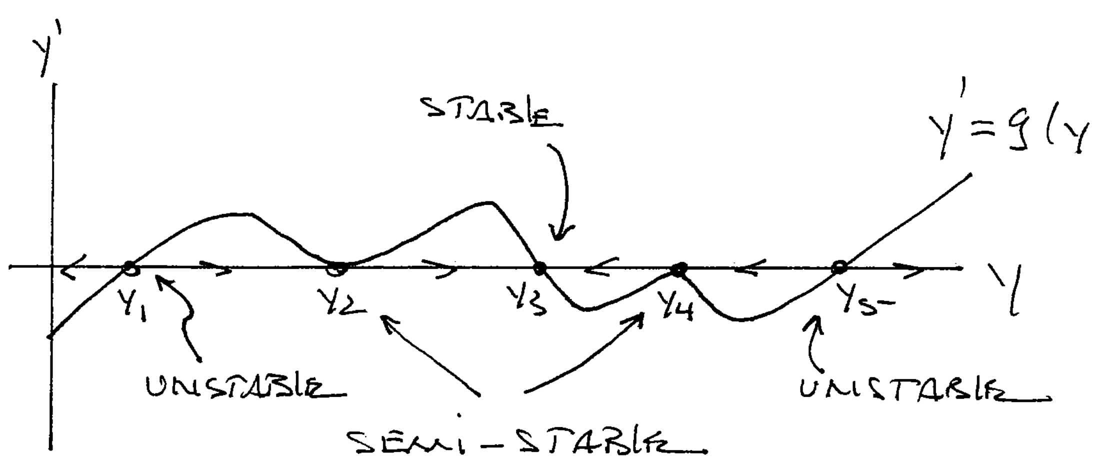
$y(x) = y_3$ is THEREFORE CALLED A <u>STABLE EQUILIBRIA</u> SINCE A SMALL CHANGE IN INITIAL CONDITIONS GIVES A NEARBY SOLUTION.

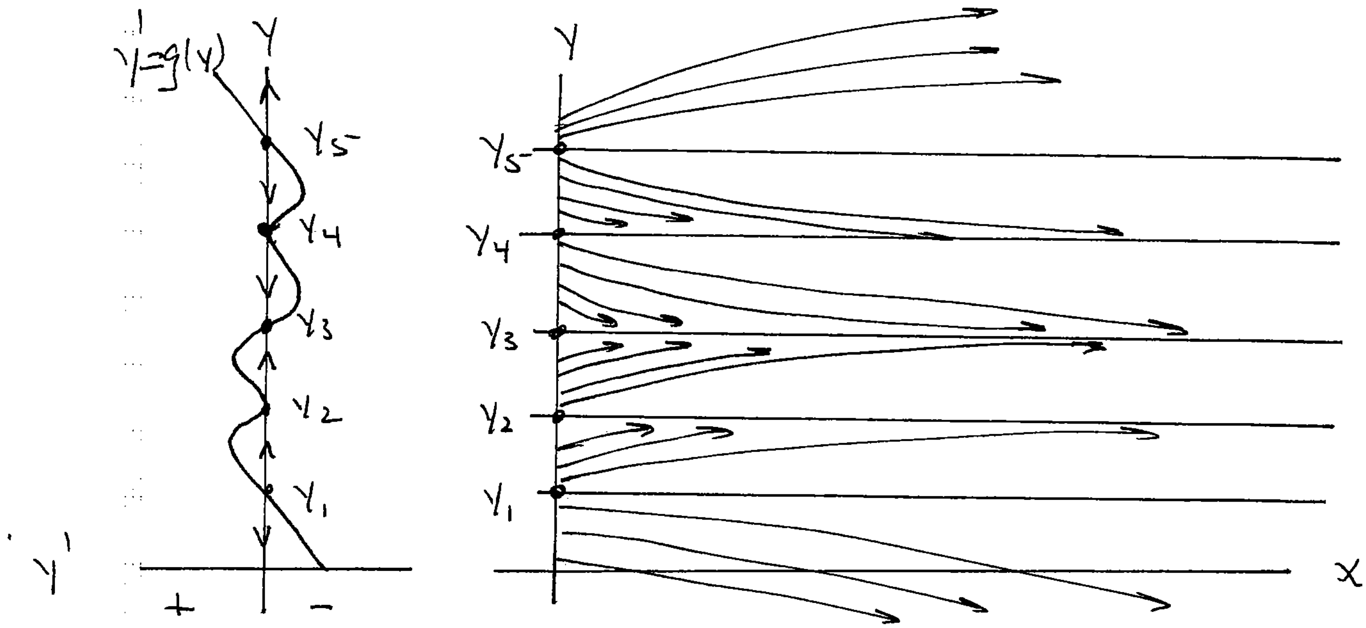
SIMILARLY $y(x) = y_2$ IS CALLED AN UNSTABLE EQUILIBRIUM.

WE CAN SEE THAT SOLUTIONS $y(x)$ WHICH START OUT NEAR A STABLE EQUILIBRIUM, STAY NEARBY FOR ALL LARGER $x$ VALUES. LIKEWISE SOLUTIONS STARTING NEAR AN UNSTABLE EQUILIBRIUM DO NOT REMAIN NEARBY FOR LARGER $x$. FOR THIS REASON, UNSTABLE EQUILIBRIA ARE NEVER OBSERVED IN NATURE.



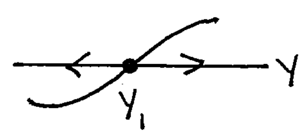

ANOTHER EXAMPLE:



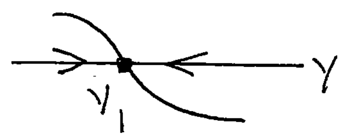



IN THIS EXAMPLE THE EQUILIBRIUM SOLUTIONS $y(x) = y_2$ AND $y(x) = y_4$ ARE CONSIDERED TO BE <u>SEMI-STABLE</u>. FOR INSTANCE, INITIAL CONDITIONS $y_0 \in (y_1, y_2)$ APPROACH $y_2$ ASYMPTOTICALLY AS $x$ INCREASES, WHILE SOLUTIONS WITH INITIAL CONDITIONS $y_0 \in (y_2, y_3)$ MOVE AWAY FROM $y_2$.

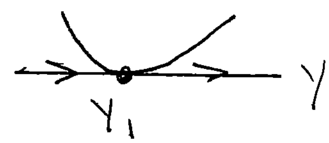
OBSERVE THAT THE ISSUE OF STABILITY CAN ALSO BE ANALYSED BY EXAMINING THE DERIVATIVE $g'(y)$ AT AN EQUILIBRIUM $y_1$.

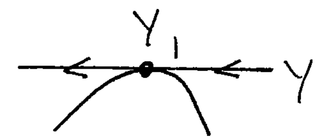
- $g'(y_1) > 0 \Rightarrow$ 


  $\Rightarrow y(x) = y_1$ IS UNSTABLE

- $g'(y_1) < 0 \Rightarrow$ 


$\Rightarrow y(x) = y_1$ IS STABLE

$g'(y_1) = 0 \Rightarrow$ 

OR 


$\Rightarrow y(x) = y_1$ IS SEMI-STABLE.

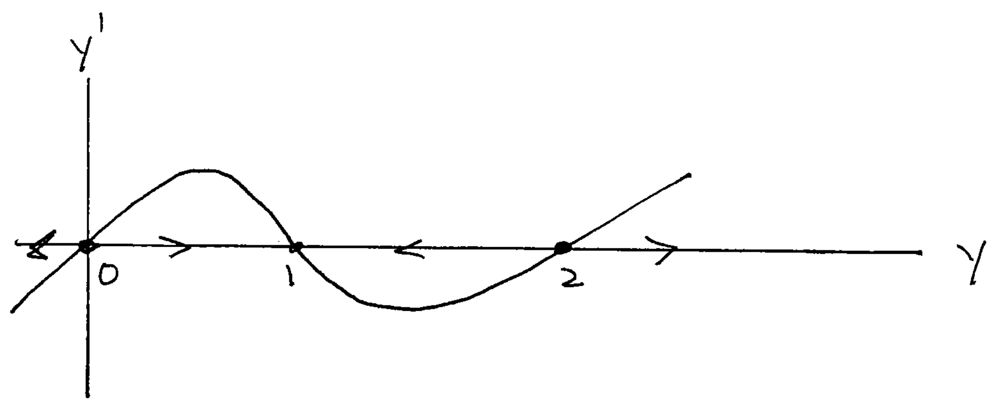
<u>Ex.</u> $\frac{dy}{dx} = y(y-1)(y-2) = \underbrace{y^3 - 3y^2 + 2y}_{g(y)}$

$g'(y) = 3y^2 - 6y + 2$

$g'(0) = 2 > 0$ UNSTABLE $\quad y(x) = 0$

$g'(1) = -1 < 0$ STABLE $\quad y(x) = 1$

$g'(2) = 2 > 0$ UNSTABLE $\quad y(x) = 2$




THE VALUES $g'(0), g'(1), g'(2)$ ARE CALLED THE EIGENVALUES OF THE EQUATION.

EX. THE LOGISTIC EQUATION

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

a) GRAPH $N' = g(N) = rN\left(1 - \frac{N}{K}\right)$

b) WHAT ARE THE EQUILIBRIUM SOLUTIONS

c.) DETERMINE THEIR STABILITY FROM THE GRAPH $N' = g(N)$

d.) COMPUTE THE EIGENVALUES $g'(N_1)$, $g'(N_2)$ ($N_1, N_2$ EQUILIBRIA) AND DETERMINE STABILITY AGAIN.

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HW 11 (DO NOT TURN IN)

(8.2.5) (P. 507) 2abc, 4abc, 6abc, 10bc