7.7.1 Taylor Polynomials

All the operations of calculus are easy, even trivial when applied to polynomials

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

For $k \in \mathbb{Z}$ non-negative we have

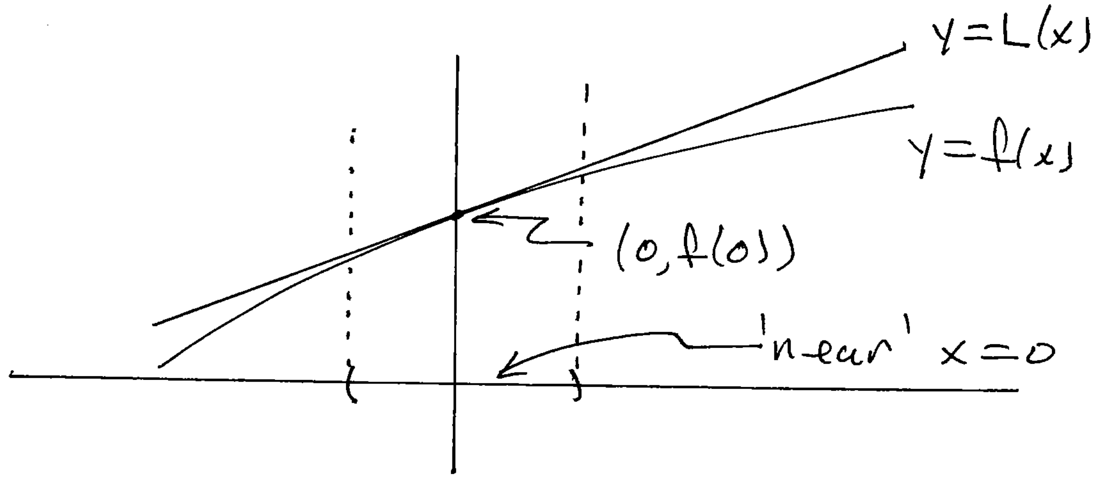
$$\frac{d}{dx}[x^k] = k x^{k-1}, \quad \int x^k dx = \frac{x^{k+1}}{k+1} + C$$

One might wish that all functions were polynomials.

The Taylor polynomials $P_n(x)$ given function $f(x)$ are an attempt to make this wish come true, at least in an approximate sense.

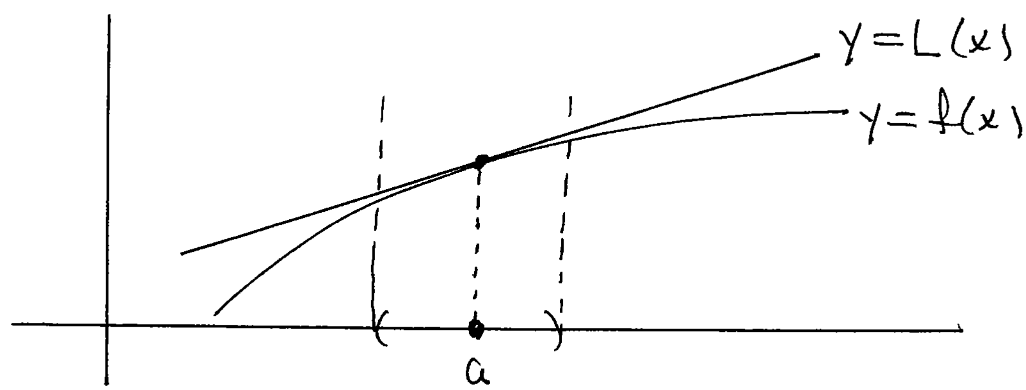
Suppose we wish to approximate a continuous function $f(x)$ near $x=0$ by a linear (i.e. 1st degree) polynomial $L(x)$.

THUS THE GRAPH $y = L(x)$ will BE A line. A REASONABLE CHOICE FOR THIS line is THE TAGENT line TO $y = f(x)$ AT THE POINT $(0, f(0))$.



$$\therefore \quad L(x) = f(0) + f'(0)\,x$$

MORE GENERALLY, WE CAN APPROXIMATE $f(x)$ NEAR ANY $x = a$ BY THE TANGENT line

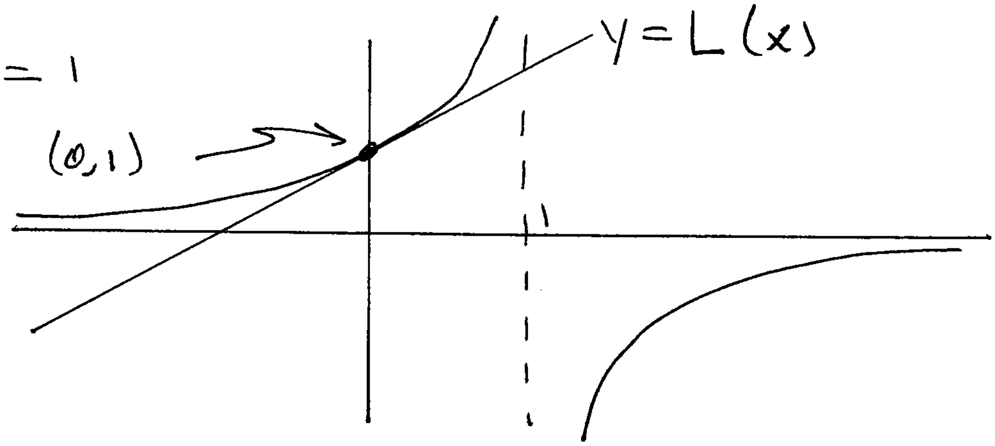


$$L(x) = f(a) + f'(a)(x-a)$$

THESE PICTURES SUGGEST THAT THE ERROR INCURRED BY APPROXIMATING $f(x)$ BY $L(x)$ IS SMALL WHEN x IS 'NEAR' a (OR NEAR 0, IN THE FIRST CASE.)

<u>Ex</u> FIND A LINEAR APPROXIMATION TO $f(x) = \frac{1}{1-x}$, FOR x NEAR 0.

$f(0) = 1$



$$f'(x) = \frac{1}{(1-x)^2}, \quad f'(0) = 1$$

$$\therefore \boxed{L(x) = 1 + x}$$

OBSERVE THAT THE LINEAR APPROXIMATION TO $f(x)$ AT $x=0$ SATISFIES

$$L(0) = f(0)$$
$$L'(0) = f'(0).$$

WE RESTRICT OUR ATTENTION TO APPROXIMATIONS NEAR $x=0$.

(TO OBTAIN APPROXIMATIONS NEAR $x=a$, REPLACE 0 BY a, AND x BY $x-a$.)

WE SEEK HIGHER ORDER APPROXIMATIONS BY REQUIRING THAT A POLYNOMIAL OF DEGREE n

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

SATISFY

$$\begin{aligned} P_n(0) &= f(0) \\ P_n'(0) &= f'(0) \\ P_n''(0) &= f''(0) \\ &\vdots \\ P_n^{(k)}(0) &= f^{(k)}(0) \\ &\vdots \\ P_n^{(n)}(0) &= f^{(n)}(0) \end{aligned}$$

i.e. WE REQUIRE THAT THE k^{TH} DERIVATIVE OF $P_n(x)$ AND $f(x)$ AT $x=0$ MATCH FOR $k = 0, 1, \ldots, n$.

OBSERVE THAT THERE ARE $n+1$ EQUATIONS IN THE $n+1$ UNKNOWN COEFFICIENTS

$$a_0, a_1, a_2, \ldots, a_n$$

THEREFORE THE APPROXIMATING POLYNOMIAL $P_n(x)$ IS DEFINED UNIQUELY.

WE CALL $P_n(x)$ THE <u>TAYLOR POLYNOMIAL OF DEGREE n ABOUT $x=0$ FOR $f(x)$</u>.

OBSERVE

$$P_n(x) = a_0 + a_1 x + \cdots + a_n x^n \qquad \therefore\ P_n(0) = a_0$$

$$P_n'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} \qquad \therefore\ P_n'(0) = a_1$$

$$P_n''(x) = 2a_2 + 3\cdot 2 a_3 x + \cdots + n(n-1) a_n x^{n-2} \qquad \therefore\ P_n''(0) = 2a_2$$

$$P_n'''(x) = 3\cdot 2 a_3 + 4\cdot 3\cdot 2 a_4 x + \cdots + n(n-1)(n-2) a_n x^{n-3}$$

$$\therefore\ P_n'''(0) = 3\cdot 2\cdot a_3$$

$\vdots$

$$P_n^{(k)}(x) = k(k-1)\cdots 2\cdot a_k + \cdots + n(n-1)\cdots(n-k+1)a_n x^{n-k}$$

$$\therefore P_n^{(k)}(0) = k!\, a_k$$

$\vdots$

$$P_n^{(n)}(x) = n(n-1)\cdots 3\cdot 2 \qquad \therefore P_n^{(n)}(0) = n!$$

THUS OUR $n+1$ EQUATIONS BECOME !

$$\begin{aligned}
a_0 &= f(0) &&\Rightarrow a_0 = f(0)\\
a_1 &= f'(0) &&\Rightarrow a_1 = f'(0)\\
2a_2 &= f''(0) &&\Rightarrow a_2 = f''(0)/2\\
3\cdot 2\, a_3 &= f'''(0) &&\Rightarrow a_3 = f'''(0)/3!\\
&\vdots\\
k!\, a_k &= f^{(k)}(0) &&\Rightarrow a_k = \frac{f^{(k)}(0)}{k!}\\
&\vdots\\
n!\, a_n &= f^{(n)}(0) &&\Rightarrow a_n = \frac{f^{(n)}(0)}{n!}
\end{aligned}$$

THUS $P_n(x)$ CAN BE WRITTEN

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!} + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

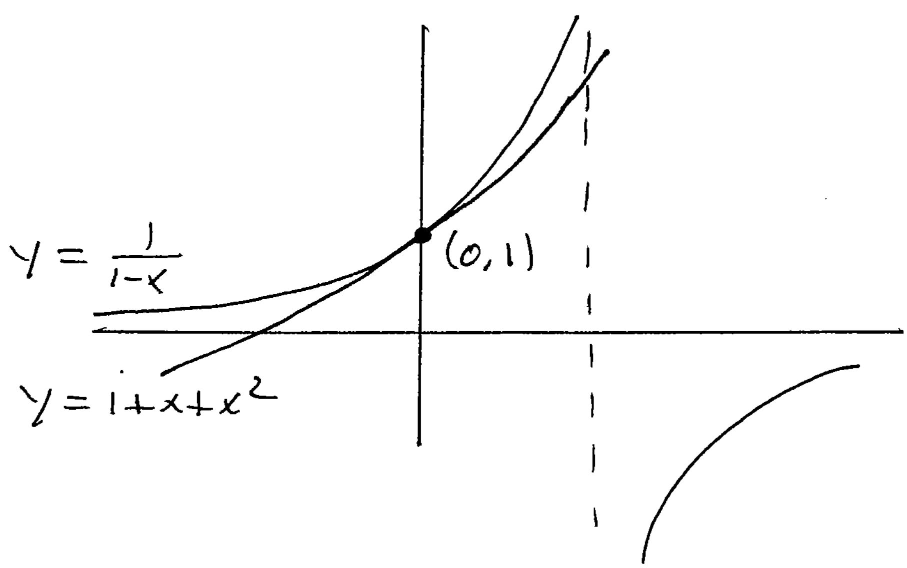
OR MORE SUCCINCTLY:

$$\boxed{P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k}$$

THIS FORMULA GIVES THE TAYLOR POLYNOMIAL FOR $f(x)$ ABOUT $x = 0$ OF DEGREE n.

Ex. $f(x) = \frac{1}{1-x}$, $n = 2$

$$P_2(x) = 1 + x + x^2$$



Ex $f(x) = \cos x$, $n = 3$

Ex $f(x) = \sqrt{1+x}$, $n = 4$

Ex $f(x) = e^{-2x}$, $n = 5$

SOMETIMES ONE CAN FIND A TAYLOR POLYNOMIAL BY SUBSTITUTING INTO A SIMPLER ONE. FOR INSTANCE ONE SHOWS EASILY THAT e^x HAS THE 5^{TH} DEGREE TAYLOR POLYNOMIAL AT $x=0$:

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

HENCE

$$e^{-2x} \approx 1 - 2x + \frac{4x^2}{2} - \frac{8x^3}{6} + \frac{16x^4}{24} - \frac{32x^5}{120}$$

$$= 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 - \frac{4}{15}x^5$$

is THE 5^{TH} DEGREE TAYLOR POLYNOMIAL FOR e^{-2x} AT $x=0$.

SOME WELL KNOWN TAYLOR POLY-NOMIALS AT $x=0$ ARE

$$e^x \approx P_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k$$

$$\cos x \approx P_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$\sin x \approx P_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$\ln(1+x) \approx P_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} x^k$$

<u>EXERCISE</u>: VERIFY THESE FORMULI

SEE LINKS TO TAYLOR POLYNOMIAL ANIMATIONS ON WEBPAGE.

<u>EXERCISE</u>: FIND TAYLOR POLYNOMIALS $(x=0)$ FOR

$$e^{x^2} \quad \text{AND} \quad \int_0^x e^{t^2}\,dt$$

BY SUBSTITUTING INTO THE ONE FOR e^x.