7.4.3 COMPARISON OF IMPROPER INTEGRALS

SOMETIMES IT IS ENOUGH TO KNOW WHETHER OR NOT A GIVEN IMPROPER INTEGRAL CONVERGES, AND ITS SPECIFIC VALUE IS NOT NEEDED.

CONVERGENCE OR DIVERGENCE CAN OFTEN BE PROVED BY COMPARING THE GIVEN IMPROPER INTEGRAL TO ONE THAT IS KNOWN TO EITHER CONVERGE OR DIVERGE

SUPPOSE $0 \leq f(x) \leq g(x)$ FOR ALL $x \in [a, \infty)$, AND SUPPOSE IT IS KNOWN THAT

$$\int_a^\infty g(x)\,dx$$

CONVERGES, WHILE AN ANTIDERIVATIVE OF $f(x)$ IS NOT EASILY FOUND, SO WE CANNOT DETERMINE THE CONVERGENCE OF

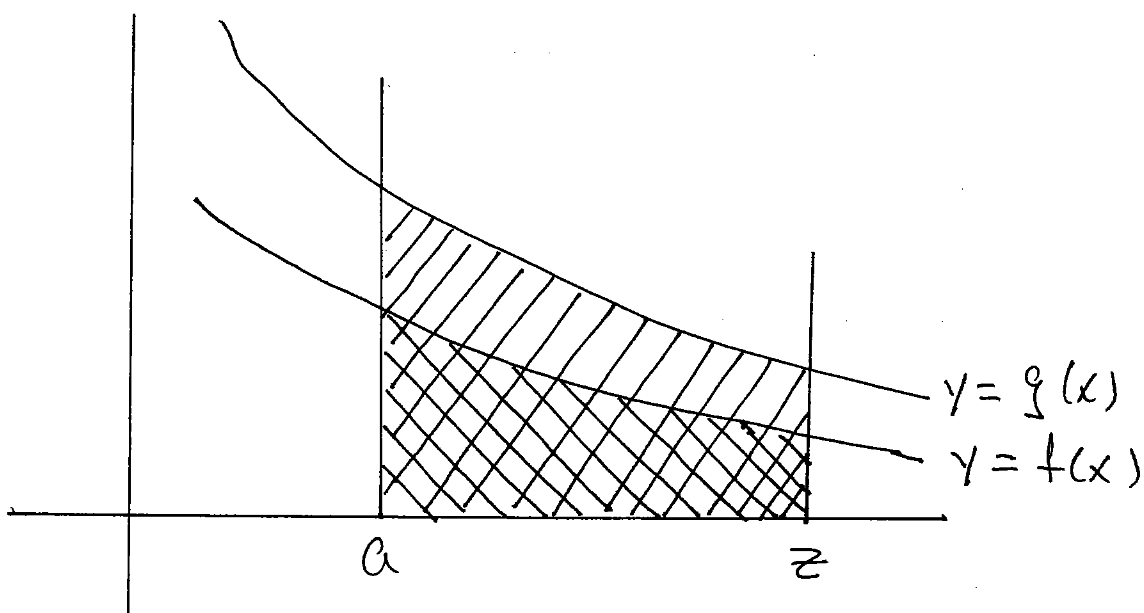
$$\int_a^\infty f(x)\,dx$$

DIRECTLY.

THEN BY PROPERTIES (5) AND (7) FROM SECTION 6.1.3 (PP. 349-352) WE HAVE

$$0 \le \int_a^z f(x)\,dx \le \int_a^z g(x)\,dx \le \int_a^\infty g(x)\,dx$$

FOR ALL $z \ge a$.



SINCE $\int_a^z f(x)\,dx$ IS AN INCREASING FUNCTION OF z, AND IS BOUNDED ABOVE BY $\int_a^\infty g(x)\,dx$, WE HAVE THAT

$$\int_0^\infty f(x)\,dx = \lim_{z\to\infty} \int_0^z f(z)\,dz$$ EXISTS AND IS FINITE, WHENCE $\int_a^\infty f(x)\,dx$ CONVERGES.

SIMILARLY IF $\int_a^\infty f(x)\,dx$ IS KNOWN TO DIVERGE, WHILE $g(x)$ IS NOT EASILY INTEGRABLE, WE GET FROM THE SAME INEQUALITY:

$$0 \le \int_a^z f(x)\,dx \le \int_a^z g(x)\,dx$$

THAT $\int_a^\infty g(x)\,dx$ MUST DIVERGE.

EX. DOES $\int_0^\infty e^{-x^2}\,dx$ CONVERGE OR DIVERGE? NOTE THERE IS NO SIMPLE WAY TO WRITE THE ANTIDERIVATIVE OF e^{-x^2}

FIRST SPLIT INTO TWO TERMS

$$\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx$$

THE FIRST TERM IS CLEARLY FINITE, SO IT ENOUGH TO DETERMINE THE CONVERGENCE OR DIVERGENCE OF

$$\int_1^\infty e^{-x^2}dx \quad .$$

BUT OBSERVE :

$$\begin{aligned} x \geq 1 &\Rightarrow x^2 \geq x \\ &\Rightarrow -x^2 \leq -x \\ &\Rightarrow e^{-x^2} \leq e^{-x} \end{aligned}$$

NOW $\displaystyle\int_1^\infty e^{-x}dx = \lim_{z \to \infty}\left(e^{-1} - e^{-z}\right) = \frac{1}{e}$

HENCE $\int_1^\infty e^{-x}dx$ CONVERGES, AND

THERFORE $\int_1^\infty e^{-x^2}dx$ ALSO CONVERGES.

$\therefore \displaystyle\int_0^\infty e^{-x^2}dx$ CONVERGES.

IN FACT ONE CAN SHOW BY MORE ADVANCED MEANS THAT

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

<u>EX.</u> SHOW $\displaystyle\int_1^\infty \frac{1}{\sqrt{1+x^2}}\,dx$ DIVERGES.

$$x \geq 1 \Rightarrow x^2 \geq 1 > \frac{1}{3} \Rightarrow 3x^2 > 1$$

$$\Rightarrow 1+x^2 < 4x^2$$

$$\Rightarrow \sqrt{1+x^2} < 2x$$

$$\Rightarrow 0 < \frac{1}{2x} < \frac{1}{\sqrt{1+x^2}}$$

AND $\displaystyle\int_1^\infty \frac{1}{2x}\,dx = \lim_{z\to\infty} \frac{1}{2}(\ln z - \ln 1) = \infty$

$\therefore \displaystyle\int_1^\infty \frac{1}{2x}\,dx$ DIVERGES $\therefore \displaystyle\int_1^\infty \frac{1}{\sqrt{1+x^2}}\,dx$ DIVERGES.

HW 7

(7.3.3) P. 424 : 24ab, 26-52 even

(7.4.4) P. 442 : 2-26 even, 30, 36ab, 38ab