7.4 IMPROPER INTEGRALS

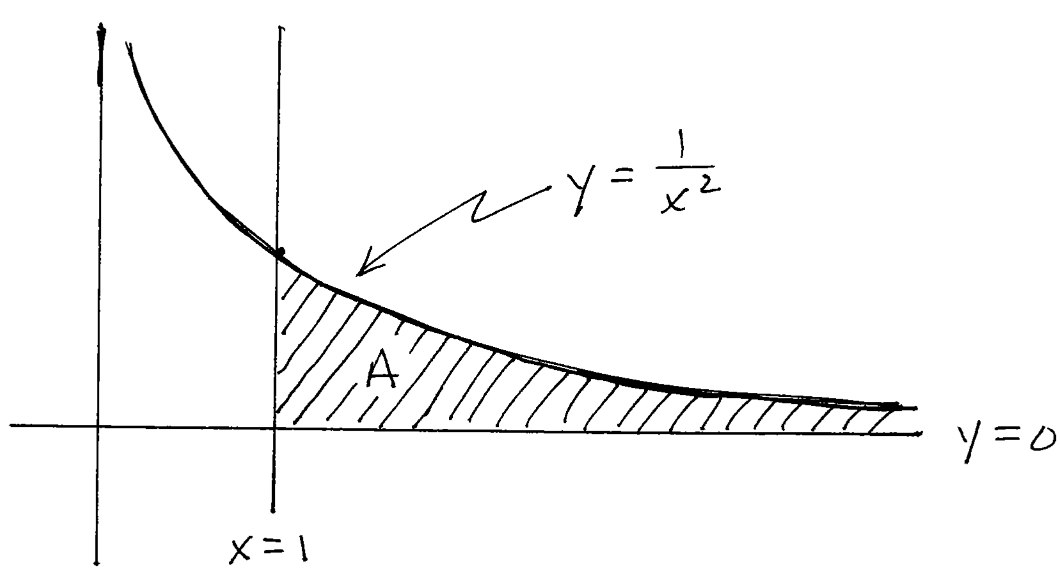
THERE ARE TWO TYPES OF IMPROPER INTEGRALS WHICH WE WILL DISCUSS.

1.) ONE OR MORE LIMITS OF INTEGRATION ARE INFINITE, i.e. THE INTERVAL OF INTEGRATION IS: $[a,\infty)$, $(-\infty,a]$, OR $(-\infty,\infty)$.

2.) THE INTEGRAND BECOMES INFINITE AT ONE OR MORE POINTS IN THE INTERVAL OF INTEGRATION.

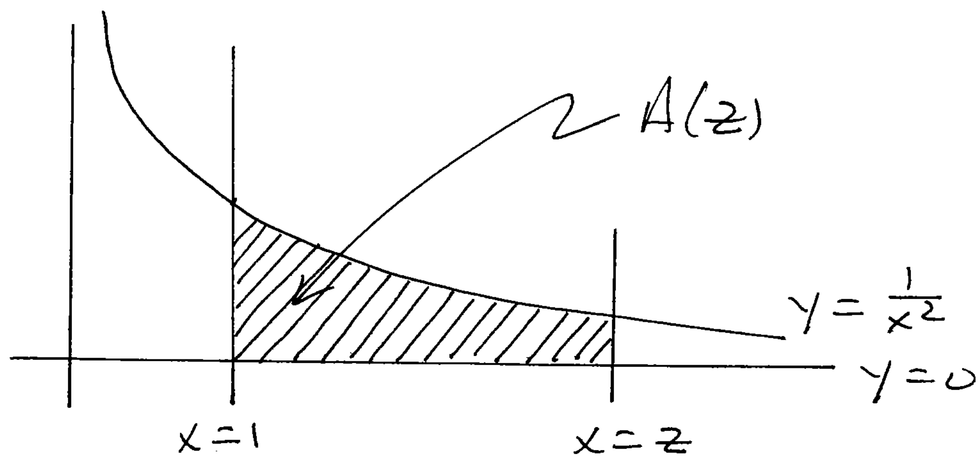
(7.4.1) INFINITE LIMITS OF INTEGRATION

<u>EX.</u> FIND THE AREA A BOUNDED BY $y = 1/x^2$, $y=0$, $x=1$.



THE REGION IN QUESTION IS UNBOUNDED, SO ITS AREA IS NOT PROPERLY DEFINED.

A RELATED PROBLEM:



FIND THE AREA $A(z)$ BOUNDED BY $y = 1/x^2$, $y = 0$, $x = 1$, AND $x = z$.

$$A(z) = \int_1^z x^{-2}\,dx = -x^{-1}\Big|_1^z$$

$$= 1 - \frac{1}{z}$$

IT IS THUS RESONABLE TO DEFINE

$$A = \lim_{z\to\infty} A(z) = \lim_{z\to\infty}\left(1 - \frac{1}{z}\right) = \boxed{1}.$$

MORE GENERALLY WE DEFINE

$$\boxed{\int_a^\infty f(x)\,dx = \lim_{z\to\infty} \int_a^z f(x)\,dx}$$

Ex. $$\int_2^\infty e^{-x}\,dx = \lim_{z\to\infty} \int_2^z e^{-x}\,dx$$

$$= \lim_{z\to\infty} -e^{-x}\Big|_2^z = \lim_{z\to\infty} \left(e^{-2} - e^{-z}\right)$$

$$= \frac{1}{e^2}$$

SIMILARLY WE DEFINE

$$\boxed{\int_{-\infty}^a f(x)\,dx = \lim_{z\to-\infty} \int_z^a f(x)\,dx}$$

Ex. $$\int_{-\infty}^5 e^{3x}\,dx = \lim_{z\to-\infty} \tfrac{1}{3}e^{3x}\Big|_z^5 = \lim_{z\to-\infty} \tfrac{1}{3}\left(e^{15} - e^{3z}\right)$$

$$= \frac{e^{15}}{3}.$$

IN BOTH DEFINITIONS, IF THE LIMIT EXISTS WE SAY THE IMPROPER INTEGRAL <u>CONVERGES</u>, OTHERWISE WE SAY IT <u>DIVERGES</u>.

WE CAN ALSO DEFINE THE INTEGRAL OF A FUNCTION OVER ALL OF $\mathbb{R}$ $= (-\infty, \infty)$ AS FOLLOWS:

<u>DEFN</u>

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx}$$

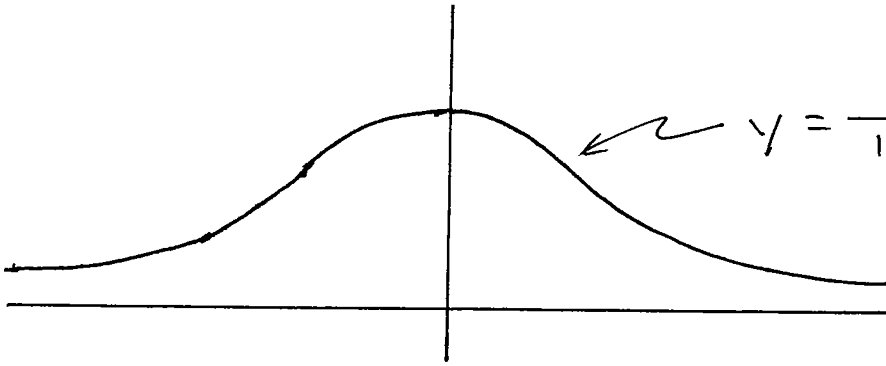
WHERE a IS ANY REAL NUMBER.

IF <u>BOTH</u> INTEGRALS ON THE RIGHT CONVERGE (INDEPENDENTLY), WE SAY THAT

$$\int_{-\infty}^{\infty} f(x)\,dx$$

<u>CONVERGES</u>, OTHERWISE IT <u>DIVERGES</u>.

EX Find the area bounded by $y = \frac{1}{1+x^2}$ and $y = 0$.



We pick $a = 0$ in the definition and write

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{z \to \infty} \left[\tan^{-1}(z) - \tan^{-1}(0)\right] = \frac{\pi}{2}$$

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{z \to -\infty} \left[\tan^{-1}(0) - \tan^{-1}(z)\right] = \frac{\pi}{2}$$

$$\therefore \quad A = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi$$

Ex. $$\int_{-\infty}^{\infty} \frac{2x}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{2x}{1+x^2}\,dx + \int_{0}^{\infty} \frac{2x}{1+x^2}\,dx$$

NOW

$$\int_{0}^{\infty} \frac{2x}{1+x^2}\,dx = \lim_{z\to\infty} \int_{0}^{z} \frac{2x}{1+x^2}\,dx$$

$$= \lim_{z\to\infty} \ln(1+x^2)\Big|_0^z$$

$$= \lim_{z\to\infty} \left[\ln(1+z^2) - 0\right] = \infty$$

THUS $\int_0^{\infty} \frac{2x}{1+x^2}\,dx$ DIVERGES, AND HENCE SO DOES $\int_{-\infty}^{\infty} \frac{2x}{1+x^2}\,dx$.

NOTE IT IS WRONG TO SAY THE TWO AREAS A_+, A_- BELOW CANCEL, SINCE THEY ARE BOTH INFINITE