7.3.2 RATIONAL FUNCTIONS

A RATIONAL FUNCTION IS A FUNCTION OF THE FORM

$$f(x) = \frac{P(x)}{Q(x)}$$

WHERE $P(x)$ AND $Q(x)$ ARE POLYNOMIALS.

EX.

$$\int \frac{1}{x^2-2x+1}\,dx = \int \frac{1}{(x-1)^2}\,dx$$

$$\left.\begin{aligned} u &= x-1 \\ du &= dx \end{aligned}\right\} = \int \frac{1}{u^2}\,du = \int u^{-2}\,du$$

$$= -u^{-1} + C = \frac{-1}{x-1} + C = \frac{1}{1-x} + C$$

EX. $\displaystyle\int \frac{1}{x^2-2x+2}\,dx$

OBSERVE x^2-2x+2 IS IRREDUCIBLE (CHECK ROOTS ARE $1 \pm i$). WE COMPLETE THE SQUARE

COMPLETE THE SQUARE

$$x^2 - 2x + 2 = (x^2 - 2x + 1) + 2 - 1$$
$$= (x-1)^2 + 1$$

THUS

$$\int \frac{1}{x^2-2x+2}\,dx = \int \frac{1}{1+(x-1)^2}\,dx$$

$u = x-1$
$du = dx$

$$= \int \frac{1}{1+u^2}\,dx$$

$$= \text{TAN}^{-1} u + C$$

$$= \text{TAN}^{-1}(x-1) + C$$

<u>EX.</u> $$\int \frac{1}{x^2-2x}\,dx = \int \frac{1}{x(x-2)}\,dx$$

CHECK: $\frac{1}{x(x-2)} = \frac{1/2}{x-2} - \frac{1/2}{x}$, SO

$$\int \frac{1}{x(x-2)}\,dx = \frac{1}{2}\int \frac{1}{x-2}\,dx - \frac{1}{2}\int \frac{1}{x}\,dx$$

$$= \frac{1}{2}\ln|x-2| - \frac{1}{2}\ln|x| + C$$

$$= \frac{1}{2}\ln\left|\frac{x-2}{x}\right| + C$$

Ex. $\displaystyle\int \frac{4x^3-6x^2+4x+5}{x^2-2x+2}\,dx$

DIVIDE THE BOTTOM INTO THE TOP

$$\begin{array}{r}
4x+2 \\
x^2-2x+2 \,\big)\,\overline{4x^3-6x^2+4x+5} \\
-\underline{(4x^3-8x^2+8x)} \\
2x^2-4x+5 \\
-\underline{(2x^2-4x+4)} \\
1
\end{array}$$

THUS

$$4x^3-6x^2+4x+5=(x^2-2x+2)(4x+2)+1$$

$$\therefore \frac{4x^3-6x^2+4x+5}{x^2-2x+2}=(4x+2)+\frac{1}{x^2-2x+2}$$

AND

$$\int \frac{4x^3-6x^2+4x+5}{x^2-2x+2}\,dx=\int\left(4x+2+\frac{1}{x^2-2x+2}\right)dx$$

$$=2x^2+2x+\tan^{-1}(x-1)+C.$$

<u>RULE</u>: IF $\deg P(x) \geq \deg Q(x)$, DIVIDE $Q(x)$ INTO $P(x)$.

WHEN THE DENOMINATOR CONTAINS LINEAR FACTORS, WE DECOMPOSE THE INTEGRAND INTO A SUM OF SIMPLER FRACTIONS. WE CALL THIS A <u>PARTIAL FRACTION DECOMPOSITION</u> (PFD) OF THE RATIONAL FUNCTION.

<u>Ex.</u> $\displaystyle\int \frac{1}{x^2+x-2}\,dx = \int \frac{1}{(x-1)(x+2)}\,dx$

PFD: $\displaystyle\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$

$\therefore \quad 1 = A(x+2) + B(x-1) \leftarrow$ THIS MUST HOLD FOR ALL x

$x = -2 \Rightarrow 1 = -3B \Rightarrow B = -1/3$

$x = 1 \Rightarrow 1 = 3A \Rightarrow A = 1/3$

THUS

$$\int \frac{1}{(x-1)(x+2)}\,dx = \int \left(\frac{1/3}{x-1} + \frac{-1/3}{x+2} \right) dx$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C$$

$$= \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C.$$

THE PFD LOOKS A LITTLE DIFFERENT WHEN THE DENOMINATOR CONTAINS REPEATED LINEAR FACTORS.

Ex. $\displaystyle\int \frac{1}{(x-1)^2(x+2)}\,dx$

PFD: $\displaystyle\frac{1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$

CLEAR ALL DENOMINATORS BY MULTIPLYING BY THE LCM: $(x-1)^2(x+2)$

$$\therefore \quad 1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2 \qquad (*)$$

SINCE THIS EQUATION MUST HOLD FOR ALL $x \in \mathbb{R}$, WE CAN CHOOSE SPECIFIC VALUES FOR x TO OBTAIN 3 EQUATIONS IN THE 3 UNKNOWNS A, B, C.

$x = 1 \Rightarrow 1 = 3B \Rightarrow B = 1/3$
$x = -2 \Rightarrow 1 = 9C \Rightarrow C = 1/9$
$x = 0 \Rightarrow 1 = -2A + 2/3 + 1/9 \Rightarrow A = -1/9$

IT IS HELPFUL TO CHOOSE VALUES OF x JUDICIOUSLY, SO THE EQUATIONS FOR A, B, C ARE EASY TO SOLVE.

ANOTHER APPROACH IS TO EXPAND THE POLYNOMIAL ON THE RHS OF (*) THEN EQUATE COEFFICIENTS RIGHT AND LEFT.

$$1 = A(x^2+x-2) + B(x+2) + C(x^2-2x+1)$$
$$1 = (A+C)x^2 + (A+B-2C)x + (-2A+2B+C)$$

THUS A, B, C SATISFY THE LINEAR SYSTEM OF EQUATIONS

$$\begin{cases} A \qquad + C = 0 \\ A + B - 2C = 0 \\ -2A + 2B + C = 1 \end{cases}$$

ONE CHECKS THAT $A = -1/9$, $B = 1/3$, $C = 1/9$ SATISFIES THIS SYSTEM.

THUS

$$\int \frac{1}{(x-1)^2(x+2)}\,dx = \int \left(\frac{-1/9}{x-1} + \frac{1/3}{(x-1)^2} + \frac{1/9}{x+2} \right) dx$$

$$= -\frac{1}{9}\ln|x-1| - \frac{1}{3}\left(\frac{1}{x-1}\right) + \frac{1}{9}\ln|x+2| + C$$

$$= \frac{1}{9}\left[\ln\left|\frac{x+2}{x-1}\right| - \frac{3}{x-1} \right] + C.$$

IN GENERAL, IF $Q(x)$ CONTAINS THE REPEATED LINEAR FACTOR

$$(ax+b)^n$$

THEN THE PFD OF $\dfrac{P(x)}{Q(x)}$ CONTAINS THE TERMS

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

(AMONGST POSSIBLY OTHER TERMS CORRESPONDING TO OTHER FACTORS.)

EX. $\dfrac{2x+5}{(x+1)^3(x+2)^2(x+1)}$

$$= \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{x+2} + \frac{E}{(x+2)^2} + \frac{F}{x+1}$$

EXERCISE: CLEAR FRACTIONS, THEN PLUG IN VALUES FOR x TO OBTAIN EQNS FOR $A-F$. DETERMINE $A-F$ THEN EVALUATE

$$\int \frac{2x+5}{(x+1)^3(x+2)^2(x+1)}\,dx$$

WHAT IF $Q(x)$ CONTAINS AN IRREDUCIBLE QUADRATIC FACTOR?

Ex. $\displaystyle\int \frac{1}{2x^2+4x+10}\,dx$

DISCRIMINANT $= 4^2 - 4\cdot 2\cdot 10 = -64 < 0$

$\therefore$ $2x^2+4x+10$ HAS NO REAL ROOTS

$\therefore$ $2x^2+4x+10$ IS IRREDUCIBLE.

COMPLETING THE SQUARE

$$2x^2+4x+10 = 2\left(x^2+2x+1\right)+10-2$$
$$= 2(x+1)^2+8$$
$$= 8\left[1+\left(\frac{x+1}{2}\right)^2\right]$$

THUS

$$\int \frac{1}{2x^2+4x+10}\,dx = \int \frac{1/8}{1+\left(\frac{x+1}{2}\right)^2}\,dx$$

$u = \frac{x+1}{2}$

$du = \frac{1}{2}dx$

$2du = dx$

$$= \frac{1}{8}\cdot 2\cdot \int \frac{1}{1+u^2}\,du$$
$$= \frac{1}{4}\tan^{-1}u + C$$
$$= \frac{1}{4}\tan^{-1}\left(\frac{x+1}{2}\right) + C.$$

IN GENERAL IF $Q(x)$ CONTAINS A REPEATED IRREDUCIBLE QUADRATIC FACTOR

$$(ax^2+bx+c)^n$$

THEN THE PFD OF $\frac{P(x)}{Q(x)}$ MUST CONTAIN THE TERMS:

$$\frac{B_1x+C_1}{ax^2+bx+c}+\frac{B_2x+C_2}{(ax^2+bx+c)^2}+\cdots+\frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

<u>EX.</u> $\displaystyle\int \frac{x^2+x+1}{(x^2+1)^2}\,dx$

$$\frac{x^2+x+1}{(x^2+1)^2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}$$

$$\begin{aligned} x^2+x+1 &= (Ax+B)(x^2+1)+Cx+D \\ &= Ax^3+Bx^2+(A+C)x+(B+D) \end{aligned}$$

$$\begin{cases} A = 0 & \Rightarrow A=0 \\ B = 1 & \Rightarrow B=1 \\ A+C = 1 & \Rightarrow C=1 \\ B+D = 1 & \Rightarrow D=0 \end{cases}$$

THUS

$$\int \frac{x^2+x+1}{(x^2+1)^2}\,dx = \int\left(\frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}\right)dx$$

$$= \tan^{-1}(x) - \frac{1}{2(x^2+1)} + C$$

EX.

$$\int \frac{3x^4+3x^3+9x^2+1}{(x-1)(x^2+x+2)^2}\,dx$$

PFD:

$$\frac{A}{x-1} + \frac{Bx+C}{(x^2+x+2)} + \frac{Dx+E}{(x^2+x+2)^2}$$

CLEARING FRACTIONS AND EQUATING TERMS YIELDS THE SYSTEM

$$\begin{cases} A + B = 3 \\ 2A + C = 3 \\ 5A + B + D = 9 \\ 4A - 2B + C - D + E = 0 \\ 4A - 2C - E = 1 \end{cases}$$

WHICH HAS THE UNIQUE SOLUTION:
$A=1$, $B=2$, $C=1$, $D=2$, $E=1$.

THUS THE INTEGRAL BECOMES

$$\int\left(\frac{1}{x-1}+\frac{2x+1}{x^2+x+2}+\frac{2x+1}{(x^2+x+2)^2}\right)dx$$

$$= \ln|x-1| + \ln|x^2+x+2| - \frac{1}{x^2+x+2} + C$$

$$= \ln|(x-1)(x^2+x+2)| - \frac{1}{x^2+x+2} + C.$$

GENERAL PROCEDURE

TO FIND $\int \frac{P(x)}{Q(x)}\,dx$

1. IF $\deg P(x) \geq \deg Q(x)$, DIVIDE $Q(x)$ INTO $P(x)$ TO OBTAIN

$$\frac{P(x)}{Q(x)} = g(x) + \frac{R(x)}{Q(x)}$$

WHERE $g(x)$ IS A POLYNOMIAL AND $\deg R(x) < \deg Q(x)$.

2. IF $\deg P(x) < \deg Q(x)$, FACTOR $Q(x)$ INTO IRREDUCIBLE LINEAR AND QUADRATIC FACTORS.

(3) DECOMPOSE $\frac{P(x)}{Q(x)}$ INTO PARTIAL FRACTIONS

(a) FOR EACH REPEATED LINEAR FACTOR $(ax+b)^n$, THE PFD MUST CONTAIN

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

(b) FOR EACH REPEATED IRREDUCIBLE QUADRATIC FACTOR $(ax^2+bx+c)^n$, THE PFD MUST CONTAIN

$$\frac{B_1x+C_1}{ax^2+bx+c} + \cdots\cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^2}.$$

(7.3.3) P. 424

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