7.2 INTEGRATION BY PARTS

RECALL THE PRODUCT RULE. LET $u = u(x)$, $v = v(x)$. THEN

$$\frac{d}{dx}[uv] = u'v + uv'$$

HENCE

$$uv = \int (u'v + uv')\,dx$$

THUS

$$\int uv'\,dx = uv - \int u'v\,dx$$

IN OTHER NOTATION:

$$\boxed{\int u \frac{dv}{dx}\,dx = uv - \int v \frac{du}{dx}\,dx}$$

IN (FORMAL) DIFFERENTIAL NOTATION:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

EX. $\int x\cos x\,dx = x\sin x - \int \sin x\,dx$

$\left\{\begin{matrix} u = x, & \frac{dv}{dx} = \cos x \\ \frac{du}{dx} = 1, & v = \sin x \end{matrix}\right\}$

$= x\sin x - (-\cos x) + C$

$= x\sin x + \cos x + C$

CHECK:

$\frac{d}{dx}\left[x\sin x + \cos x + C\right] = x\cos x + \cancel{\sin x} - \cancel{\sin x} + 0$

$= x\cos x$

OFTEN DIFFERENTIAL NOTATION IS USED WHEN DOING THE INTEGRATION BY PARTS SUBSTITUTIONS.

$\left\{\begin{matrix} u = x & dv = \cos x\,dx \\ du = dx & v = \sin x \end{matrix}\right.$

$$\int \underbrace{x}_{u}\,\underbrace{\cos x\,dx}_{dv} = \underbrace{x}_{u}\,\underbrace{\sin x}_{v} - \int \underbrace{\sin x}_{v}\,\underbrace{dx}_{du}$$

$= \dots$

$= x\sin x + \cos x + C.$

Ex. $\int x \ln x \, dx \quad = \frac{1}{2}x^2 \ln x - \int \frac{1}{2} x \, dx$

$$\left\{ \begin{matrix} u = \ln x & dv = x \\ du = \frac{1}{x} dx & v = \frac{1}{2}x^2 \end{matrix} \right\}$$

$$= \boxed{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}$$

EXERCISE: CHECK BY DIFFERENTIATION

HOW TO DECIDE WHAT PART OF THE INTEGRAND TO ASSIGN AS u, AND WHAT PART AS dv.

GOAL: ASSIGN u & dv IN SUCH A WAY THAT

$$\int v \, du$$

IS SIMPLER THAN

$$\int u \, dv$$

IF ONE FINDS THE NEW INTEGRAND TO BE MORE COMPLICATED THAN THE ORIGINAL, LOOK FOR ANOTHER SUBSTITUTION.

<u>EX.</u> $\displaystyle\int x e^x\,dx = xe^x - \int e^x\,dx$

$$\left\{\begin{array}{ll} u = x & dv = e^x dx \\ du = dx & v = e^x \end{array}\right\} = xe^x - e^x + C$$

LET'S SEE WHAT HAPPENS WITH THE 'WRONG' SUBSTITUTION

$u = e^x \quad dv = x\,dx$
$du = e^x dx \quad v = \frac{1}{2}x^2$

SO

$$\int \underline{x e^x}\,dx = \frac{1}{2}x^2 e^x - \int \underline{\frac{1}{2}x^2 e^x}\,dx$$

THIS IS WORSE THAN THIS

WITH SOME EXPERIENCE, ONE CAN LOOK AHEAD AND IDENTIFY A GOOD SUBSTITUTION.

SOMETIMES WE CAN USE INTEGRATION BY PARTS MULTIPLE TIMES.

<u>Ex.</u> $\int x^3 e^x \, dx$

$$\begin{cases} u = x^3 & dv = e^x \, dx \\ du = 3x^2 \, dx & v = e^x \end{cases}$$

$$= x^3 e^x - 3 \int x^2 e^x \, dx$$

$$\begin{cases} u = x^2 & dv = e^x \, dx \\ du = 2x \, dx & v = e^x \end{cases}$$

$$= x^3 e^x - 3 \left[x^2 e^x - 2 \int x e^x \, dx \right]$$

$$= x^3 e^x - 3x^2 e^x + 6 \int x e^x \, dx$$

$$= x^3 e^x - 3x^2 e^x + 6 \left(x e^x - e^x \right) + C$$

$$= \boxed{(x^3 - 3x^2 + 6x - 6) e^x + C}$$

SOMETIMES THE SUBSTITUTION IS FAR FROM OBVIOUS.

Ex $\int \ln x \, dx = x \ln x - \int 1 \, dx$

$$\left\{ \begin{matrix} u = \ln x & dv = dx \\ du = \frac{1}{x} dx & v = x \end{matrix} \right\} = \boxed{x \ln x - x + C}$$

Ex. $\int \tan^{-1} x \, dx = x \tan^{-1} x - \int \frac{x}{1+x^2} dx$

$$\left\{ \begin{matrix} u = \tan^{-1} x & dv = dx \\ du = \frac{1}{1+x^2} dx & v = x \end{matrix} \right\} = x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx$$

$$= \boxed{x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C}$$

Ex. $\overbrace{\int e^x \sin x \, dx}^{I}$ $\quad \left\{ \begin{matrix} u = \sin x & dv = e^x dx \\ du = \cos x \, dx & v = e^x \end{matrix} \right.$

$$= e^x \sin x - \int e^x \cos x \, dx \quad \left\{ \begin{matrix} u = \cos x & dv = e^x dx \\ du = -\sin x \, dx & v = e^x \end{matrix} \right.$$

$$= e^x \sin x - \left(e^x \cos x - \int -e^x \sin x \, dx \right)$$

$$= e^x (\sin x - \cos x) - \underbrace{\int e^x \sin x \, dx}_{I}$$

$$\therefore \quad 2I = e^x(\sin x - \cos x)$$

$$I = \frac{1}{2}e^x(\sin x - \cos x)$$

$$\therefore \int e^x \sin x \, dx = \frac{1}{2}e^x(\sin x - \cos x) + C$$

EXERCISE: CHECK THIS.

<u>Ex.</u> $\int x^2 \ln x^2 \, dx$ $\left\{ \begin{array}{ll} u = \ln x^2 & dv = x^2 dx \\ du = \frac{2}{x} dx & v = \frac{1}{3}x^3 \end{array} \right.$

$$= \frac{1}{3}x^3 \ln x^2 - \frac{2}{3}\int x^2 dx$$

$$= \boxed{\frac{1}{3}x^3 \ln x^2 - \frac{2}{9}x^3 + C}$$

$$\int_1^2 x^2 \ln x^2 \, dx = \left(\frac{1}{3}x^3 \ln x^2 - \frac{2}{9}x^3\right)\Bigg|_1^2$$

$$= \boxed{\frac{16}{3}\ln 2 - \frac{14}{9}}$$

I.B.P. FOR DEFINITE INTEGRALS:

$$\int_a^b uv'\,dx = uv\Big|_a^b - \int_a^b vu'\,dx$$

EX. $\int_1^4 \ln\sqrt{x}\,dx = x\ln\sqrt{x}\Big|_1^4 - \int_1^4 \frac{1}{2}\,dx$

$u = \ln\sqrt{x}$ $\quad dv = dx$

$du = \frac{1}{2x}dx$ $\quad v = x$

$$= (4\ln 2 - 0) - \frac{1}{2}x\Big|_1^4$$

$$= 4\ln 2 - \frac{1}{2}(4-1) = \boxed{4\ln 2 - \frac{3}{2}}$$

(7.2.1) P. 414

2-34 even, 44, 48