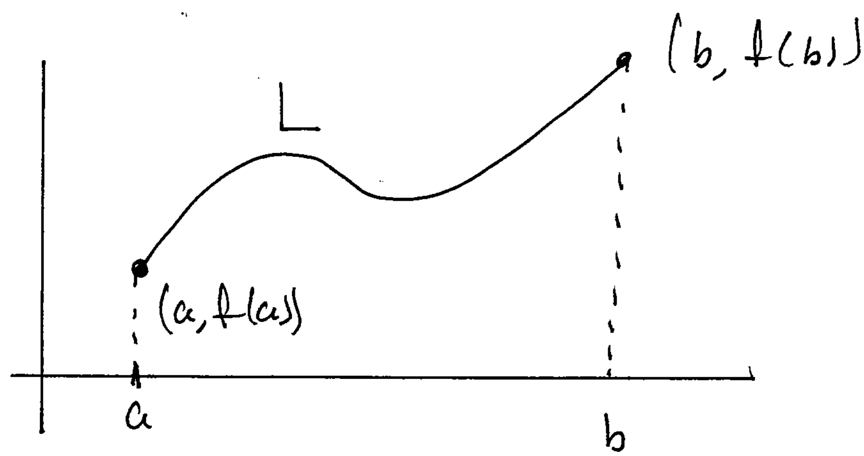
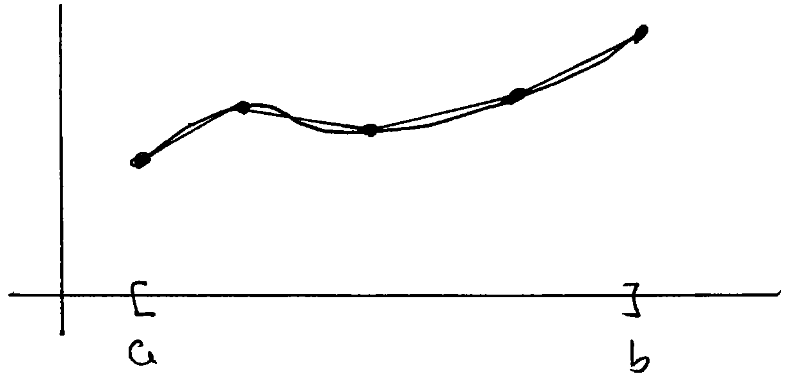
(SKIP 6.3.4)

6.3.5 RECTIFICATION OF CURVES.

CONSIDER THE PROBLEM OF FINDING THE LENGTH OF A CURVE $y = f(x)$ FROM $x = a$ TO $x = b$.



WE APPROXIMATE THE CURVE BY A SEQUENCE OF STRAIGHT LINE SEGMENTS, THEN ADD UP THE LENGTHS OF THESE SEGMENTS. IF THE SEGMENTS ARE SUFFIENTLY SMALL, THIS SUM SERVES AS A GOOD APPROXIMATION TO THE LENGTH L.



LET $P = [x_0, x_1, \ldots, x_n]$ BE A PARTITION OF $[a, b]$ INTO n SUBINTERVALS

$$a = x_0 < x_1 < \ldots < x_n = b$$

AND LET

$$y_i = f(x_i) \quad \text{FOR } 0 \leq i \leq n$$

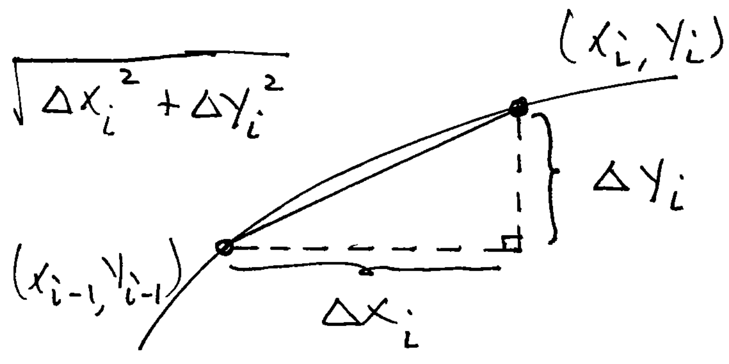
GIVING A SEQUENCE

$$y_0, y_1, \ldots, y_n$$

OF POINTS ON THE Y-AXIS. DEFINE

$$\Delta x_i = x_i - x_{i-1} \quad \text{AND} \quad \Delta y_i = y_i - y_{i-1}$$

FOR $1 \leq i \leq n$.



THE i^{TH} LINE SEGMENT JOINS POINTS (x_{i-1}, y_{i-1}) AND (x_i, y_i) AND HAS LENGTH

$$\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

BY THE PYTHAGOREAN THEOREM.

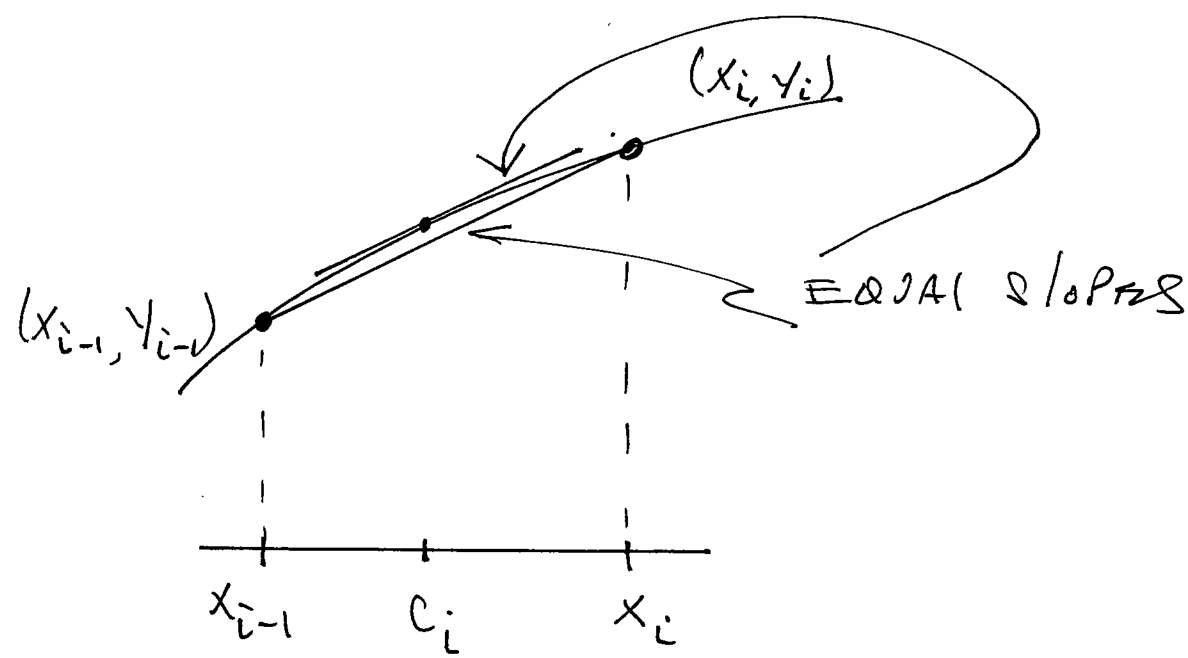
THUS L IS APPROXIMATED BY THE SUM

$$L_p = \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$= \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \cdot \Delta x_i$$

NOW $\frac{\Delta y_i}{\Delta x_i}$ IS THE SLOPE OF THE i^{TH} LINE SEGMENT. THE MVT SAYS THERE EXISTS A $c_i \in [x_{i-1}, x_i]$ SUCH THAT

$$f'(c_i) = \frac{\Delta y_i}{\Delta x_i}$$



THUS

$$L_P = \sum_{i=1}^{n} \sqrt{1 + \left(f'(c_i)\right)^2} \ \Delta x_i$$

THIS APPROXIMATION GETS BETTER AS THE PARTITION GETS FINER. TAKING THE LIMIT AS $\|P\| \to 0$ WE HAVE

$$L = \int_a^b \sqrt{1 + (f'(x))^2} \, dx$$

SOMETIMES THIS IS WRITTEN AS

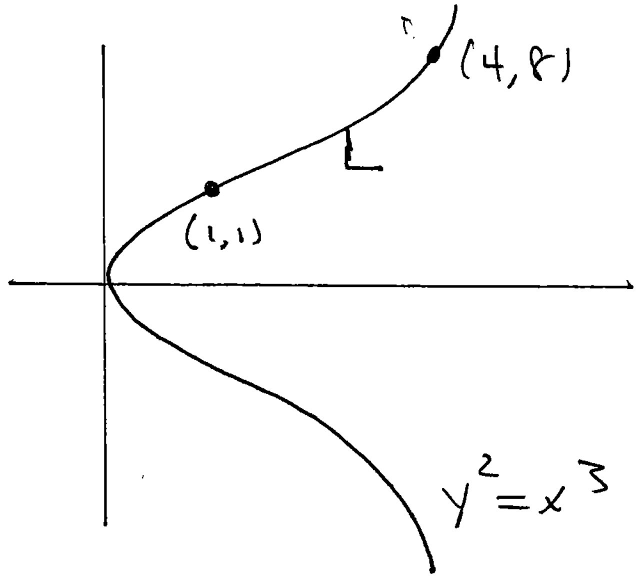
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

OR AS

$$L = \int_a^b \sqrt{(dx)^2 + (dy)^2}$$

THE SYMBOL $\sqrt{(dx)^2 + (dy)^2}$ IS CALLED THE <u>ARC LENGTH DIFFERENTIAL</u> AND IS SOMETIMES DENOTED ds.

Ex. FIND THE LENGTH OF THE CURVE $y^2 = x^3$ FROM $(1,1)$ TO $(4,8)$



$$y = x^{3/2} \Rightarrow \frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}.$$

$$L = \int_1^4 \sqrt{1+\left(\tfrac{3}{2}x^{1/2}\right)^2}\,dx = \int_1^4 \sqrt{1+\tfrac{9}{4}x}\;dx$$

OBSERVE $\displaystyle \frac{d}{dx}\left[\frac{4}{9}\cdot\frac{2}{3}\left(1+\frac{9}{4}x\right)^{3/2}\right]$

$$= \frac{4}{9}\cdot\frac{2}{3}\cdot\frac{3}{2}\left(1+\frac{9}{4}x\right)^{1/2}\cdot\frac{9}{4}$$

$$= \sqrt{1+\tfrac{9}{4}x}$$

THUS

$$L = \frac{8}{27}\left(1+\frac{9}{4}x\right)^{3/2}\Bigg|_{1}^{4}$$

$$= \frac{8}{27}\left[10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]$$

Hw3 6.3.6 (P. 392)

2-16 even

18, 22, 24, 26, 28, 32

54, 56, 58, 60, 62