MATH 11B
CALCULUS WITH APPLICATIONS
FALL 2007

5.8 ANTIDERIVATIVES

WE BEGIN WITH A SIMPLE EXAMPLE.

<u>EX.</u> DETERMINE A FUNCTION $y = F(x)$ SATISFYING THE CONDITION

$$\frac{dy}{dx} = 3x^2$$

OR EQUIVALENTLY

$$F'(x) = 3x^2$$

OBSERVE THAT IF $y = x^3$, THEN $\frac{dy}{dx} = 3x^2$ SO $F(x) = x^3$ SOLVES THE PROBLEM

NOTE ALSO THAT $y = x^3 + 2$, AND $y = x^3 + 50$ SATISFY THE GIVEN CONDITION EQUALLY WELL. IN FACT, ANY FUNCTION OF THE FORM

$$y = x^3 + C$$

WHERE C IS AN ARBITRARY CONSTANT, SATISFIES

$$\frac{dy}{dx} = 3x^2$$

THUS THERE ARE INFINITELY MANY VALID ANSWERS TO OUR ORIGINAL QUESTION.

DEFN.

A FUNCTION $F(x)$ IS SAID TO BE AN ANTIDERIVATIVE OF $f(x)$ ON AN INTERVAL $I \subseteq \mathbb{R}$ IFF

$$F'(x) = f(x) \quad \text{FOR ALL} \quad x \in I$$

WE SEE THAT AN ANTIDERIVATIVE OF $f(x)$ IS ANY SOLUTION TO A DIFFERENTIAL EQUATION OF THE FORM

$$\frac{dy}{dx} = f(x),$$

AND IN GENERAL, SUCH AN EQUATION HAS INFINITELY MANY SOLUTIONS.

EX FIND ANTIDERIVATIVES OF THE FOLLOWING FUNCTIONS.

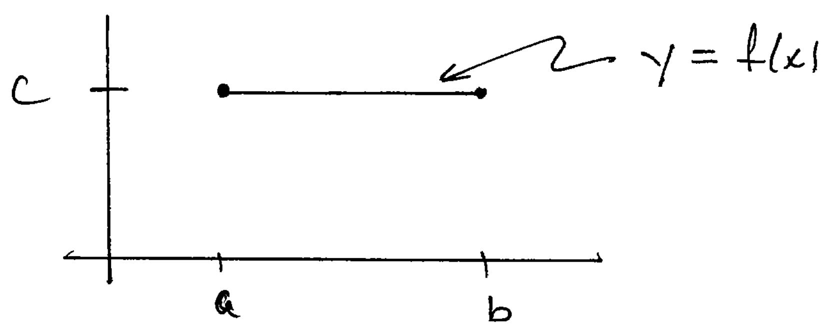
$$12x^5,\ 35x^6,\ 40x^7,\ x^{1/2},\ x^2+2x-1$$
$$e^x,\ e^{2x},\ \cos x,\ \sin x,\ \sec^2 x$$

NOTE EACH OF THE ABOVE HAS INFINITELY MANY ANTIDERIVATIVES.

RECALL COROLLARY 2 FROM SEC. 5.1

COROLLARY (P. 258)
IF $f(x)$ IS CONTINUOUS ON $[a,b]$, DIFFERENTIABLE ON (a,b), AND $f'(x)=0$ FOR ALL $x \in (a,b)$, THEN $f(x)$ IS CONSTANT ON $[a,b]$.

i.e. THERE EXISTS $C \in \mathbb{R}$ SUCH THAT $f(x)=C$ FOR ALL $x \in [a,b]$.



THIS COROLLARY FOLLOWS FROM THE MEAN VALUE THEOREM. (P. 255)

NOTICE THIS IS THE CONVERSE OF THE ASSERTION THAT

$$f(x) = \text{CONST.} \implies f'(x) = 0$$

THIS COROLLARY HAS THE FOLLOWING CONSEQUENCE.

COROLLARY (P. 326).
IF $F(x)$ AND $G(x)$ ARE TWO ANTIDERIVATIVES OF $f(x)$ ON I, THEN $F(x)$ AND $G(x)$ DIFFER BY A CONSTANT.

I.e. THERE EXISTS $C \in \mathbb{R}$ SUCH THAT

$$G(x) = F(x) + C \quad \text{FOR ALL } x \in I.$$

THUS TO FIND ALL ANTIDERIVATIVES, ONE NEED FIND ONLY ONE ANTIDERIVATIVE

PROOF
OUR HYPOTHESES SAY THAT

$$G'(x) = f(x) = F'(x),$$

WHENCE

$$0 = G'(x) - F'(x) = \frac{d}{dx}\left[G(x) - F(x)\right],$$

FOR ALL $x \in I$. APPLYING THE EARLIER COROLLARY TO THE FUNCTION $G(x) - F(x)$, WE OBTAIN

$$G(x) - F(x) = C$$

FOR SOME CONSTANT C. ///.

EX. FIND THE GENERAL ANTIDERIVATIVE (i.e. ALL ANTIDERIVATIVES) OF

$$f(x) = x^4 - 3x^2 + 1$$

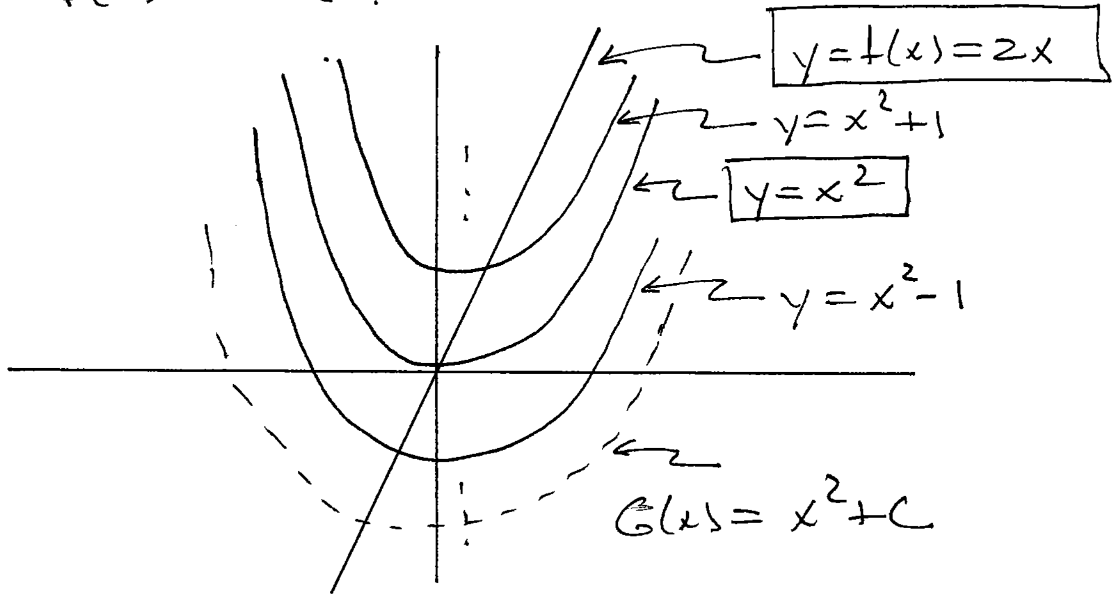
WE SEE THAT $F(x) = \frac{1}{5}x^5 - x^3 + x$ IS A PARTICULAR ANTIDERIVATIVE. THUS ALL ANTIDERIVATIVES ARE OF THE FORM

$$G(x) = \frac{1}{5}x^5 - x^3 + x + C$$

HERE SOME PARTICULAR ANTIDERIVATIVES:

FUNCTION	ANTIDERIVATIVE
$k f(x)$	$k F(x)$
$f(x) + g(x)$	$F(x) + G(x)$
$x^n \quad (n \neq -1)$	$\frac{1}{n+1} x^{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^{ax}	$\frac{1}{a} e^{ax}$
$\sin(ax)$	$-\frac{1}{a}\cos(ax)$
$\cos(ax)$	$\frac{1}{a}\sin(ax)$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax)$

EX GRAPH THE FAMILY OF ANTIDERIVATIVES OF $f(x) = 2x$.



ALL ANTIDERIVATIVES ARE OBTAINED GRAPHICALLY BY TRANSLATING $F(x) = x^2$ VERTICALLY BY AN ARBITRARY DISTANCE C.

OFTEN, WE WISH TO SELECT A PARTICULAR ANTIDERIVATIVE WHICH PASSES THROUGH A SPECIFIC POINT (x_0, y_0). THIS IS CALLED AN <u>INITIAL VALUE PROBLEM</u> (IVP)

SUCH PROBLEMS ARE USUALLY STATED AS A DIFFERENTIAL EQUATION

<u>EX.</u> $\frac{dy}{dx} = 2x$ AND $y = -5$ WHEN $x = 4$

<u>SOLN:</u> $y = x^2 - 21$

Ex. FIND A FUNCTION $N(t)$ SATISFYING

$$\frac{dN}{dt} = \frac{1}{2\sqrt{t}} \quad (t > 0)$$

AND $N(0) = 20$

WE HAVE $N'(t) = \frac{1}{2} t^{-\frac{1}{2}}$ SO THAT

$$N(t) = t^{1/2} + C = \sqrt{t} + C$$

THUS

$$20 = N(0) = \sqrt{0} + C$$

$$\therefore \quad C = 20$$

$$\therefore \quad \boxed{N(t) = \sqrt{t} + 20}\ .$$

HW 5.8.1 (P. 330)

2 - 20 even, 38-48 even,
50-60 even, 66