CMPS 201 4-7-10

2nd handout: some common fcns

- floor : $\lfloor x \rfloor = (\text{greatest int} \leq x)$
- ceiling : $\lceil x \rceil = (\text{least int} \geq x)$

i.e. $x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$

<u>equivalently</u>: If $x \in \mathbb{R}$ and $N \in \mathbb{Z}$

then

(1) $N = \lfloor x \rfloor$ iff $N \leq x < N+1$

(2) $N = \lceil x \rceil$ iff $N-1 < x \leq N$

Lemma 1: Let $x \in \mathbb{R}$, $a, b \in \mathbb{Z}$. then

(1) $a \le x < b$ iff $a \le \lfloor x \rfloor < b$.

(2) $a < x \le b$ iff $a < \lceil x \rceil \le b$.

Proof of (1):

(i) $a \le x \Rightarrow a \le \lfloor x \rfloor$, since among all ints which are $\le x$, $\lfloor x \rfloor$ is largest,

(iii) $a \le \lfloor x \rfloor \Rightarrow a \le x$, since $\lfloor x \rfloor \le x$,

(ii) $x < b \Rightarrow \lfloor x \rfloor < b$, since $\lfloor x \rfloor \le x$.

(iv) $\lfloor x \rfloor < b \Rightarrow x < b$ since by (i) (contrapositive & Replacing a by b):

$b \le x \Rightarrow b \le \lfloor x \rfloor$ //

(Contrapositive of $P \Rightarrow Q$ is

$\neg Q \Rightarrow \neg P$

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg Q$	$\neg Q \Rightarrow \neg P$
0	0	1	1	1.	1
0	1	1	1	0	1
1	0	0	0	1	0
1	1	1	0	0	1

$\therefore \; P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$.)

<u>Exercise</u>: You do $(\Leftarrow)$.

Lemma 2

Let $x \in \mathbb{R}$, $m \in \mathbb{Z}^+$. then

(1) $\left\lfloor \frac{\lfloor x \rfloor}{m} \right\rfloor = \left\lfloor \frac{x}{m} \right\rfloor$.

(2) $\left\lceil \frac{\lceil x \rceil}{m} \right\rceil = \left\lceil \frac{x}{m} \right\rceil$.

Proof of (1):

Let $N = \left\lfloor \frac{\lfloor x \rfloor}{m} \right\rfloor$. we have

$\therefore \quad N \le \frac{\lfloor x \rfloor}{m} < N+1$

$\therefore \quad mN \le \lfloor x \rfloor < m(N+1)$

$\therefore \quad mN \le x < m(N+1)$

$\therefore \quad N \le \frac{x}{m} < N+1$

$\therefore \quad N = \left\lfloor \frac{x}{m} \right\rfloor$. ///

Lemma 3: Let $a, b \in \mathbb{Z}^+$, $n \in \mathbb{Z}^+$. Then

(1) $\left\lfloor \dfrac{\lfloor n/a \rfloor}{b} \right\rfloor = \left\lfloor \dfrac{n}{ab} \right\rfloor$.

(2) $\left\lceil \dfrac{\lceil n/a \rceil}{b} \right\rceil = \left\lceil \dfrac{n}{ab} \right\rceil$.

Proof of (1): Let $x = n/a$, $m = b$ in Lemma 2. ///.

Exercise
You prove (2).

Exercise: Let $n \in \mathbb{Z}$. Show

ⓐ $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = n$.

ⓑ $\lceil \frac{n}{2} \rceil = \lfloor \frac{n+1}{2} \rfloor$.

ⓒ $\lfloor \frac{n}{2} \rfloor = \lceil \frac{n-1}{2} \rceil$.

Stirling's Formula: let $n \in \mathbb{Z}^+$.

Then

$$\boxed{n! = \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)}$$

some fcn in the class $\Theta\left(\frac{1}{n}\right)$.

Could say:

$$\frac{n!}{\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n} - 1 = \Theta\left(\frac{1}{n}\right)$$

Corollary:

(1) $n! = o(n^n)$

(2) $n! = \omega(b^n)$ for any $b > 0$.

(3) $\log(n!) = \Theta(n \log n)$

<u>Proof of (1)</u>:

$$\frac{n!}{n^n} = \frac{\sqrt{2\pi n}\cdot\frac{\cancel{n^n}}{e^n}\cdot\left(1+\Theta\left(\frac{1}{n}\right)\right)}{\cancel{n^n}}$$

$$= \sqrt{2\pi}\cdot\frac{n^{1/2}}{e^n}\cdot\left(1+\Theta\left(\frac{1}{n}\right)\right) \longrightarrow 0 .$$

<u>Proof of (3)</u>:

take log of both sides of Stirling's:

$$\log n! = \log\sqrt{2\pi} + \frac{1}{2}\log n + n\log n - n\log e + \log\left(1+\Theta\left(\frac{1}{n}\right)\right)$$

$$\therefore \frac{\log n!}{n\log n} = \underset{\downarrow\atop 0}{\frac{\log\sqrt{2\pi}}{n\log n}} + \underset{\downarrow\atop 0}{\frac{1}{2n}} + \underset{\downarrow\atop 1}{1} - \underset{\downarrow\atop 0}{\frac{\log e}{\log n}} + \underset{\downarrow\atop 0}{\frac{\log(\quad)}{n\log n}}$$

$\therefore$

$$\frac{\log n!}{n\log n} \longrightarrow 1 \quad \text{and} \quad 0 < 1 < \infty . \quad ///.$$

Exercise:

Prove $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$.

where $\binom{m}{k} = \left(\# \text{ k-ssets of an n-set}\right)$

$(0 \le k \le m)$ $\qquad = \dfrac{m!}{k!\,(m-k)!}$

$$\boxed{\binom{m}{k+1} = \binom{m-1}{k-1} + \binom{m-1}{k}}$$

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & \vdots & & & \vdots & & & \vdots & & &
\end{array}
$$

Exercise:

determine $a > 0$ such that

$$\binom{3n}{n} = \Theta\left(\frac{a^n}{\sqrt{n}}\right).$$

Induction Proofs:

Let $P(n)$ be a propositional function of $n \in \mathbb{Z}$. use induction to prove stmts. of the form:

$$\forall n \geq n_0 : P(n)$$

Outline: two steps

I. Base step (case):

Prove directly that $P(n_0)$ is true.

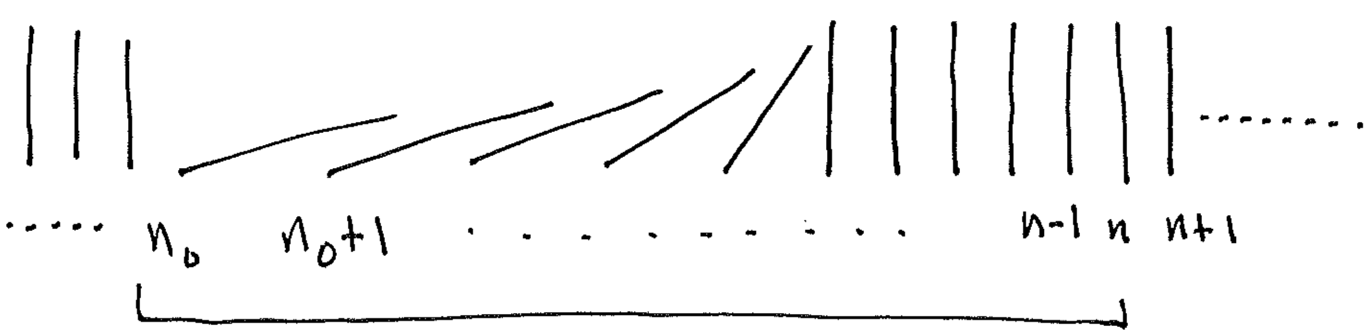
induction hypothesis

IIa. Induction step:

Prove $\forall n \geq n_0 : P(n) \Rightarrow P(n+1)$

Conclude: $\forall n \geq n_0 : P(n)$

<u>Domino Analogy</u>:



Variations on II.:

II b. Prove $\forall n > n_0 : \underbrace{P(n-1)}_{\text{ind. Hyp.}} \rightarrow P(n)$

II a & II b are called <u>weak induction</u>

<u>1st Principle of M.I</u>

II c. & II d. below are called strong induction or 2nd P.M.I.

II c. Prove $\forall n \geq n_0$

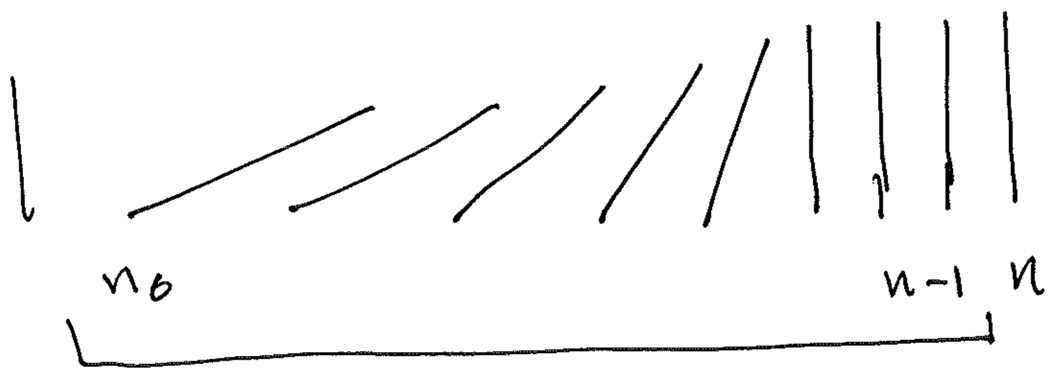
$$\underbrace{\left[P(n_0) \wedge P(n_0+1) \wedge \dots \wedge P(n)\right]}_{\text{ind. hyp.}} \rightarrow P(n+1)$$

$$\left[\forall k : n_0 \leq k \leq n : P(k)\right] \longrightarrow P(n+1)$$

II d. Prove $\forall n > n_0$: Induction hypothesis

$$\left[P(n_0) \land P(n_0+1) \land \dots \land P(n-1)\right] \Rightarrow P(n)$$

$$\left[\forall k : n_0 \le k < n : P(k)\right] \Rightarrow P(n)$$



use only

- II a II b (weak)
- II d (strong)

nobody uses II c.