CMPS 201 4-5-10

__Recall:__

__Defn:__ $f(n) = o(g(n))$ iff

$$\forall c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq f(n) < c \cdot g(n)$$

__Lemma:__ $f(n) = o(g(n))$ iff

$$\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = 0 .$$

__Exercise:__ $o(g(n)) \cap \Omega(g(n)) = \phi$.

__Defn:__ $\omega(g(n))$ is the set

$$\{ f(n) \mid \forall c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq c \cdot g(n) < f(n) \}$$

Recall $\Omega(g(n))$ was the set

$$\{f(n) \mid \exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq c \cdot g(n) \leq f(n)\}$$

we see that $\omega(g(n)) \subseteq \Omega(g(n))$.

Lemma: $f(n) = \omega(g(n))$ IFF

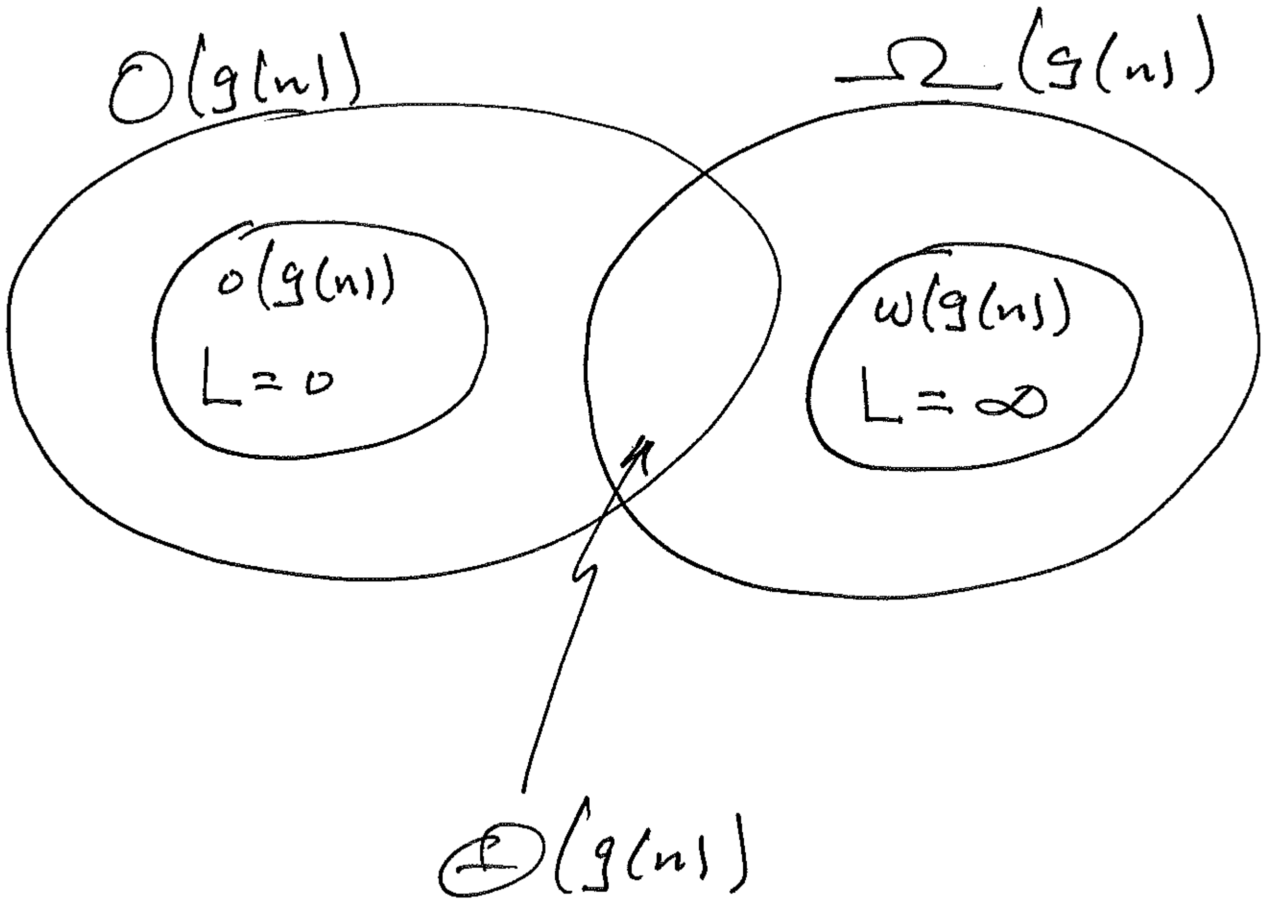
$$\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = \infty .$$

EXERCISE:

(a). Prove lemma.

(b) Prove $\omega(g(n)) \cap O(g(n)) = \phi$

Priliminary Picture



let $L = \lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right)$.

<u>Thm.</u> If $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = L$ where

$0 \leq L < \infty$, then $f(n) = O(g(n))$.

<u>warning:</u> converse is false!

<u>Proof:</u> limit stmt says:

$\forall \varepsilon > 0, \exists n_0 > 0, \forall n \geq n_0 : \left| \frac{f(n)}{g(n)} - L \right| < \varepsilon$.

Since this is true for any $\varepsilon > 0$ we can pick $\varepsilon = 1$, then there must exist $n_0 > 0$ s.t. $\forall n \geq n_0$:

$$\left| \frac{f(n)}{g(n)} - L \right| < 1$$

$$\therefore \quad \underbrace{-1 < \frac{f(n)}{g(n)} - L < 1}_{\text{focus on this}}$$

$\therefore \forall n \geq n_0$ we have

$$f(n) < (L+1)\, g(n)$$

Also can take n_0 large enough that $\forall n \geq n_0$:

$$0 \leq f(n) < (L+1)\, g(n)$$

Let $c = L+1$ in defn of $O(g(n))$.

observe $c > 0$ since $L \geq 0$.

$$\therefore \quad f(n) = O(g(n))$$

As claimed. ///

<u>THm</u>. If $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = L$, where $0 < L \le \infty$, then $f(n) = \Omega(g(n))$.

<u>warning</u>: Converse is false.

<u>Proof</u>: we have from a known Calculus THm that

$$\lim_{n\to\infty}\left(\frac{g(n)}{f(n)}\right) = L'$$

where $L' = \frac{1}{L}$, hence $0 \le L' < \infty$.

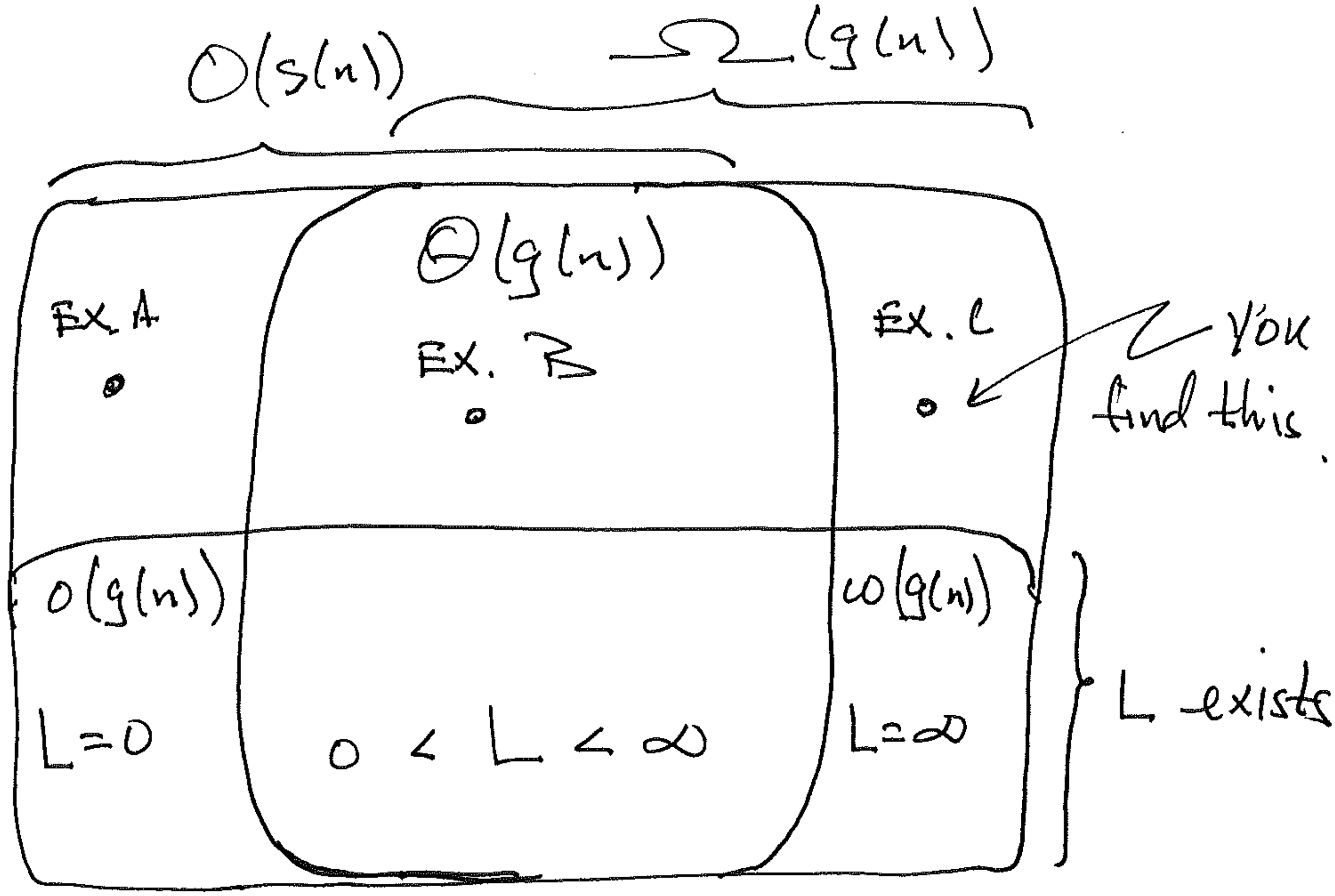
By last thm. $g(n) = O(f(n))$. By Previous THm $f(n) = \Omega(g(n))$.

///.

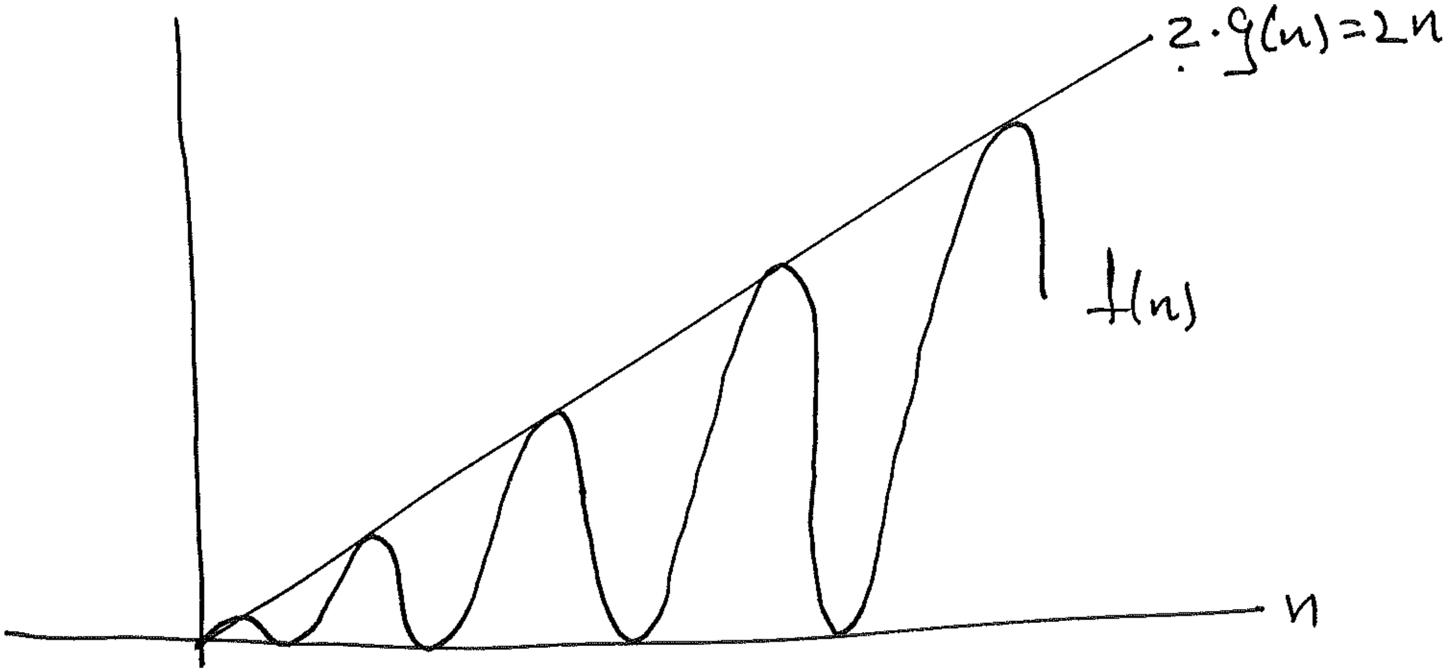
Thm: If $\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = L$ where $0 < L < \infty$, then $f(n) = \Theta(g(n))$.

PROOF: EXERCISE.

more Refined Picture:



Ex A. Let $g(n) = n$, $f(n) = (1 + \sin(n))n$



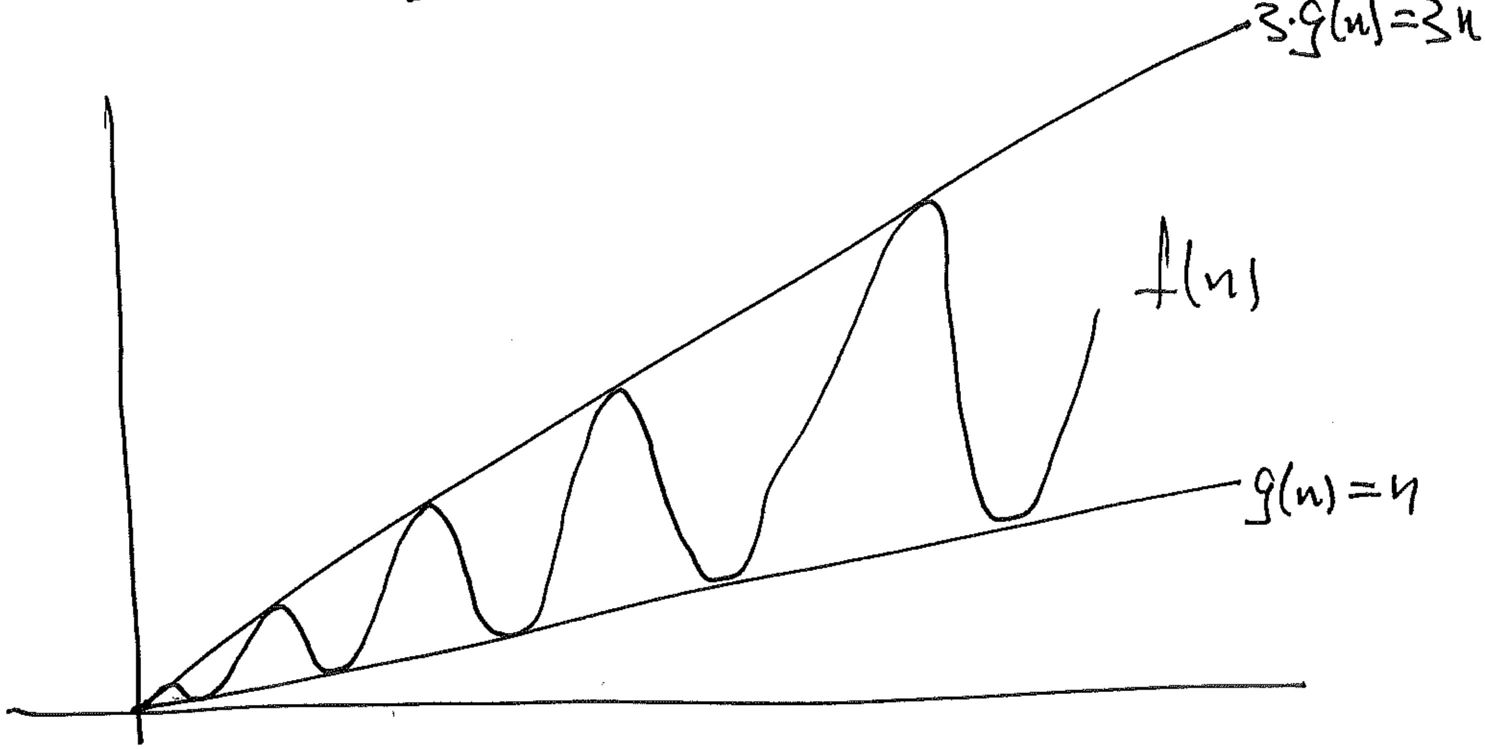
obviously $f(n) = O(g(n))$.

Also $f(n) \neq \Omega(g(n))$.

Observe $\dfrac{f(n)}{g(n)} = 1 + \sin(n)$

which has no limit as $n \to \infty$.

Ex. B. Let $g(n) = n$, $f(n) = (2 + \sin(n)) \cdot n$



$\therefore \; f(n) = \Theta(g(n))$

But $\quad \dfrac{f(n)}{g(n)} = 2 + \sin(n)$

Has no limit.

<u>Exercise</u> Dream up Ex. C.

Recall earlier exercise:

Let $a, b \in \mathbb{R}$, $b > 0$, show

$$(n+a)^b = \Theta(n^b)$$

<u>Soln</u>:

$$\lim_{n\to\infty} \frac{(n+a)^b}{n^b} = \lim_{n\to\infty} \left(\frac{n+a}{n}\right)^b$$

$$= \lim_{n\to\infty} \left(1+\frac{a}{n}\right)^b$$

$$= \left(\lim_{n\to\infty} \left(1+\frac{a}{n}\right)\right)^b \quad \left\{ \begin{array}{l} \text{since } x^b \\ \text{is cont.} \end{array} \right.$$

$$= 1^b = 1$$

and $0 < 1 < \infty$.

$\therefore (n+a)^b = \Theta(n^b)$ by last Thm.

Pg. 8 Exercise at top

(d) given $\alpha > 0$, $\beta > 0$:

$$n^{\alpha} = \begin{cases} o(n^{\beta}) & \text{iff} \quad \alpha < \beta \\ \Theta(n^{\beta}) & \text{iff} \quad \alpha = \beta \\ \omega(n^{\beta}) & \text{iff} \quad \alpha > \beta \end{cases}$$

Soln.

$$\frac{n^{\alpha}}{n^{\beta}} = n^{\alpha-\beta} \xrightarrow[\text{as } n \to \infty]{} \begin{cases} 0 & \text{iff} \quad \alpha < \beta \\ 1 & \text{iff} \quad \alpha = \beta \\ \infty & \text{iff} \quad \alpha > \beta \end{cases}.$$

Recall analogy:

$$f(n) = O(g(n)) \quad \sim \quad x \leq y$$

$$f(n) = \Omega(g(n)) \quad \sim \quad x \geq y$$

$$f(n) = \Theta(g(n)) \quad \sim \quad x = y$$

$$f(n) = o(g(n)) \quad \sim \quad x < y$$

$$f(n) = \omega(g(n)) \quad \sim \quad x > y$$

(e). let $a > 0$, $b > 0$. then

$$a^n = \begin{cases} o(b^n) & \text{iff} \quad a < b \\ \Theta(b^n) & \text{iff} \quad a = b \\ \omega(b^n) & \text{iff} \quad a > b \end{cases}$$

soln to (es):

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n \xrightarrow[\text{as } n\to\infty]{} \begin{cases} 0 & \text{iff } a<b \\ 1 & \text{iff } a=b \\ \infty & \text{iff } a>b \end{cases}$$

what about $n^{1/2}$ vs. $\ln(n)$

$$\lim_{n\to\infty} \frac{n^{1/2}}{\ln(n)} = \lim_{n\to\infty} \frac{\frac{1}{2} n^{-1/2}}{1/n}$$

$$= \frac{1}{2} \cdot \lim_{n\to\infty} \sqrt{n} = \infty$$

$\therefore \; n^{1/2} = \omega(\ln(n))$.

(g) show $\quad f(n) + o(f(n)) = \Theta(f(n))$

stands for some fcn $h(n) = o(f(n))$.

i.e. $f(n) + h(n) = \Theta(f(n))$ when $h(n) = o(f(n))$

why?

$$\frac{f(n)+h(n)}{f(n)} = 1 + \left(\frac{h(n)}{f(n)}\right) \longrightarrow 1$$

$$\downarrow$$

0 since $h(n) = o(f(n))$

since $0 < 1 < \infty$ we have

$$f(n) + h(n) = \Theta(f(n)).$$

last exercise on p. 8:

(c) compare $n^{\ln(\ln n)}$ to $2^{(\ln n)^2}$

want:

$$\lim_{n\to\infty}\left(\frac{n^{\ln(\ln n)}}{2^{(\ln n)^2}}\right) = 0$$

ans.

$n^{\ln(\ln n)} = o\left(2^{(\ln n)^2}\right)$.

let $y_n = \dfrac{n^{\ln(\ln n)}}{2^{(\ln n)^2}}$. then

$$\ln y = \ln(\ln n)\cdot\ln(n) - (\ln n)^2\cdot\ln 2$$

$$= \ln(n)\cdot\left[\ln(\ln n) - \ln 2\cdot\ln n\right] \to -\infty$$

check: $\ln(\ln n) = o(\ln n)$

$\therefore$ $\ln y_n \to -\infty$ as $n\to\infty$.

$\therefore$ $y_n \to 0$.

2nd HANDOUT : Logarithms

Let $x, a, b \in \mathbb{R}$, $x > 0$, $a > 1$, $b > 1$.

Then $\log_a(x)$ satisfies :

$$a^{\log_a(x)} = x \quad \text{and} \quad \log_a(a^x) = x$$

So

$$x = a^{\log_a(x)} = \left(b^{\log_b a}\right)^{\log_a(x)} = b^{\log_b a \cdot \log_a x}$$

So

$$\log_b(x) = \log_b a \cdot \log_a(x) \qquad (*)$$

i.e.

$$\log_b(n) = \underbrace{\log_b a}_{\text{const}} \cdot \log_a(n)$$

Thus

$$\log_b(n) = \Theta(\log_a(n))$$

Also by ⊛

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

Also by ⊛

$$a^{\log_b(x)} = a^{\log_a(x)\cdot\log_b(a)} = \left(a^{\log_a x}\right)^{\log_b a}$$

$$\therefore \quad a^{\log_b(x)} = x^{\log_b(a)}$$