CMPS 201 4-28-10

Quicksort(A, p, r)

1.) if $p < r$

2.) $q \leftarrow$ Partition(A, p, r)

3.) Quicksort(A, p, q - 1)

4.) Quicksort(A, q + 1, r)

Partition(A, p, r)

returns q and arranges $A[p \ldots r]$ s.t.

$$\underbrace{A[p \ldots (q-1)]}_{\text{not sorted}} \le A[q] \le \underbrace{A[(q+1) \ldots r]}_{\text{not sorted}}$$

↑ Pivot element (A[q])

Partition(A, p, r)

1.) $i \leftarrow p-1$

2.) for $j \leftarrow p$ to $(r-1)$

3.) if $A[j] \leq A[r]$

4.) $i \leftarrow i+1$

5.) $A[i] \leftrightarrow A[j]$

6.) $A[i+1] \leftrightarrow A[r]$

7.) return $(i+1)$

Ex.

i	j=p						r-1	r
[ ]	8	6	1	3	7	2	5	(4) ← Pivot
		j						
	8	6	1	3	7	2	5	(4)
			j					
	8	6	1	3	7	2	5	(4)
	i			j				
	1	6	8	3	7	2	5	(4)
		i			j			
	1	3	8	6	7	2	5	(4)
		i				j		
	1	3	8	6	7	2	5	(4)
			i				j	
	1	3	2	6	7	8	5	(4)
				i+1				
	(1	3	2)	(4)	(7	8	5	6)

Rmks

loop 2-5 maintains invariants:

(1) $i \le j < r$

(2) $A[p \ldots i] \le A[r]$

(3) $A[r] \le A[(i+1) \ldots (j-1)]$

(4) $A[j \ldots (r-1)]$ not yet processed.

may be $\ge A[r]$ or $\le A[r]$.

Exercise:

- Prove these invariants.

Correctness of Partition( ) follows from these invariants.

Runtime:

will do worst & AVG. cases

note: Partition( ) does length[A[p...r]] − 1 = r − p comparisons in all cases.

worst case occurs when A[p...r] is already sorted!

Pivot ↓

$A[p....(r-1)] \leq A[r] \leq ....\text{empty}....$

(sorted)

↓

$q = r$

let T(n) = worst case # of comps of Quicksort(A, 1, n).

$$T(n) = \begin{cases} 0 & n = 0, 1 \\ T(n-1) + (n-1) & n \geq 2 \end{cases}$$

$$\begin{aligned} T(n) &= (n-1) + T(n-1) \\ &= (n-1) + (n-2) + T(n-2) \\ &= (n-1) + (n-2) + (n-3) + T(n-3) \\ &\vdots \\ &= \sum_{i=1}^{k} (n-i) + T(n-k) \end{aligned}$$

choose k st $n-k=1$, i.e. $k = n-1$

$$\therefore T(n) = \sum_{i=1}^{n-1} (n-i) = \sum_{i=1}^{n-1} i$$

so $T(n) = \frac{n(n-1)}{2} = \frac{1}{2}n^2 - \frac{1}{2}n = \Theta(n^2)$

Average case:

Assume that all $n!$ permutations of input Array $A[1...n]$ are equally likely, i.e. each has probability $\frac{1}{n!}$.

let $t(n)$ = average # comp done by Quicksort$(A, 1, n)$. so

$$t(n) = \frac{\sum_{\text{all permutations}} (\text{\# comp performed on that perm.})}{n!}$$

Our assumption implies that the pivot element $A[q]$ is equally likely to be placed in any of positions $1 \ldots n$. i.e. return value q of Partition( ) has probability $\frac{1}{n}$ of being any of #'s from $1 \ldots n$. i.e.

$$P(q=i) = \frac{1}{n} \quad (\text{for } i = 1, \ldots n),$$

<u>note</u>:

- $\text{length}[A[1 \ldots (q-1)]] = q-1$
- $\text{length}[A[(q+1) \ldots n]] = n-q$

- Partition(A, 1, n) does $n-1$ comparisons.

$$\therefore\ t(n) = \frac{\sum_{q=1}^{n} \left((n-1) + t(q-1) + t(n-q) \right)}{n}$$

$$= (n-1) + \left(\sum_{q=1}^{n} t(q-1) + \sum_{q=1}^{n} t(n-q) \right) \frac{1}{n}$$

<u>note</u>: $t(0) = 0 = t(1)$, $t(2) = 1$.

$$\therefore\ t(n) = (n-1) + \frac{1}{n} \left(\sum_{q=1}^{n-1} t(q) + \sum_{q=1}^{n-1} t(n-q) \right)$$

$$\therefore\ \boxed{t(n) = (n-1) + \frac{2}{n} \cdot \sum_{q=1}^{n-1} t(q)}$$

define

$$x_n = \sum_{q=1}^{n-1} t(q)$$

$\therefore \quad x_{n+1} - x_n = t(n)$

$\therefore \quad x_{n+1} - x_n = (n-1) + \frac{2}{n} \cdot x_n$

$\therefore \quad x_{n+1} - \left(1 + \frac{2}{n}\right) x_n = n - 1$

$$x_{n+1} - \left(\frac{n+2}{n}\right) x_n = n - 1$$

multiply by : $\frac{1}{(n+1)(n+2)}$

$$\frac{x_{n+1}}{(n+1)(n+2)} - \frac{x_n}{n(n+1)} = \frac{n-1}{(n+1)(n+2)}$$

$$= \frac{-2}{n+1} + \frac{3}{n+2}$$

replace n by j:

$$\frac{x_{j+1}}{(j+1)(j+2)} - \frac{x_j}{j(j+1)} = \frac{3}{j+2} - \frac{2}{j+1}$$

sum $j=1$ to $n-1$:

$$\sum_{j=1}^{n-1}\left(\frac{x_{j+1}}{(j+1)(j+2)} - \frac{x_j}{j(j+1)}\right) = \sum_{j=1}^{n-1}\left(\frac{1}{j+2} + \frac{2}{j+2} - \frac{2}{j+1}\right)$$

$$\therefore \frac{x_n}{n(n+1)} - \cancel{\frac{x_1}{2}} = \sum_{j=3}^{n+1}\frac{1}{j} + \left(\frac{2}{n+1} - 1\right)$$

recall $x_1 = 0$

$$\therefore \frac{x_n}{n(n+1)} = \left(\sum_{j=1}^{n}\frac{1}{j}\right) + \frac{1}{n+1} - 1 - \frac{1}{2} + \frac{2}{n+1} - 1$$

define $H_n = \sum_{j=1}^{n} \frac{1}{j}$ (n^{th} Harmonic number)

$$\therefore \frac{x_n}{n(n+1)} = \frac{3}{n+1} - \frac{5}{2} + H_n$$

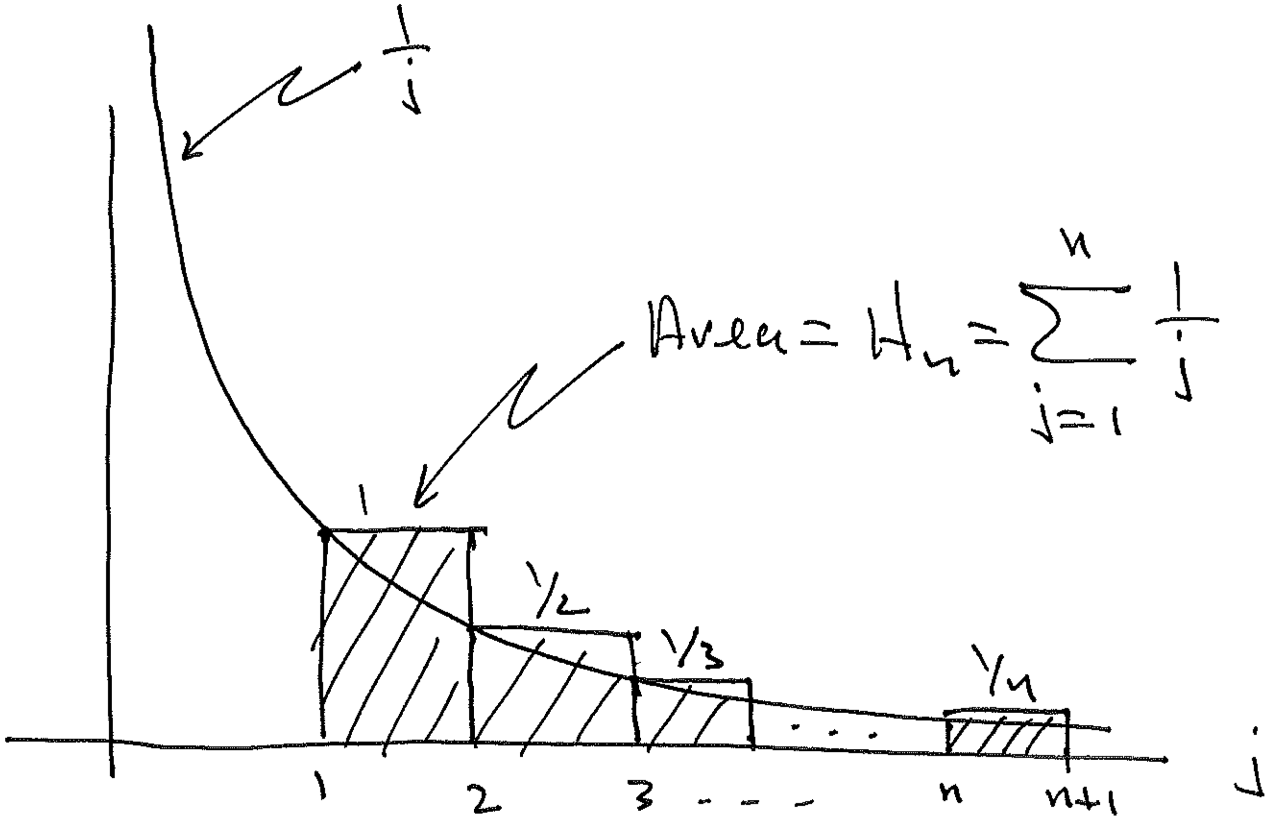
$$\therefore \boxed{x_n = n(n+1)H_n + 3n - \frac{5}{2}(n)(n+1)}$$

$$\therefore t(n) = (n-1) + \frac{2}{n} x_n$$

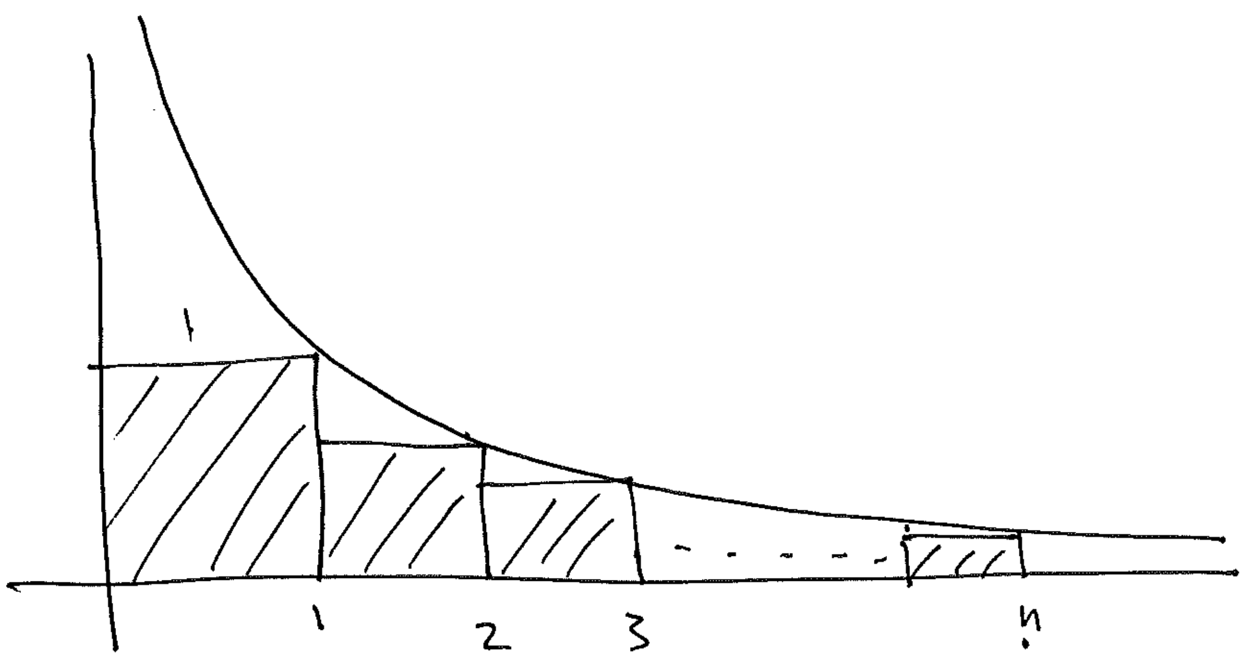
$$= (n-1) + \frac{2}{n}\left(\qquad \right)$$

$$\therefore t(n) = (n-1) + 2(n+1)H_n + 6 - 5(n+1)$$

$$\therefore \boxed{t(n) = 2(n+1)H_n - 4n}$$



$$\int_1^{n+1} \frac{1}{x}\,dx \le H_n \le 1 + \int_1^{n} \frac{1}{x}\,dx$$



$\therefore \quad \underbrace{\ln(n+1)}_{\Omega(\ln(n))} \leq H_n \leq \underbrace{1 + \ln(n)}_{O(\ln(n))}$

$\therefore \quad H_n = \Theta(\log n)$

$\therefore \quad t(n) = 2(n+1)\,\Theta(\log n) - 4n$

$\therefore \quad \boxed{t(n) = \Theta(n \log n)}.$