CMPS 201 4-19-10 

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Ex. $T(n) = \begin{cases} 1 & n = 1 \\ T(\lfloor n/2 \rfloor) + n^2 & n \geq 2 \end{cases}$.

$$
\begin{aligned}
T(n) &= n^2 + T(\lfloor n/2 \rfloor) \\
&= n^2 + \lfloor n/2 \rfloor^2 + T\left(\left\lfloor \lfloor n/2 \rfloor / 2 \right\rfloor\right) \\
&= n^2 + \lfloor n/2 \rfloor^2 + T(\lfloor n/2^2 \rfloor) \\
&= n^2 + \lfloor n/2 \rfloor^2 + \lfloor n/2^2 \rfloor^2 + T(\lfloor n/2^3 \rfloor) \\
&\vdots
\end{aligned}
$$

$$T(n) = \sum_{i=0}^{k-1} \left\lfloor \frac{n}{2^i} \right\rfloor^2 + T\left(\left\lfloor \frac{n}{2^k} \right\rfloor\right)$$

Determine $k$ s.t. $\left\lfloor \frac{n}{2^k} \right\rfloor = 1$.

$$1 \le \frac{n}{2^k} < 2$$

$$2^k \le n < 2^{k+1}$$

$$k \le \lg n < k+1$$

$$\therefore \boxed{k = \lfloor \lg n \rfloor}$$

for this value of $k$:

$$T(n) = \sum_{i=0}^{k-1} \left\lfloor \frac{n}{2^i} \right\rfloor^2 + 1$$

so

$$T(n) \le \sum_{i=0}^{k-1} \left( \frac{n}{2^i} \right)^2 + 1 \quad \left\{ \begin{array}{l} \text{since} \\ \lfloor x \rfloor \le x \end{array} \right.$$

$$= n^2 \left( \sum_{i=0}^{k-1} \left(\frac{1}{4}\right)^i \right) + 1$$

$$= n^2 \left( \frac{1 - \left(\frac{1}{4}\right)^k}{1 - \left(\frac{1}{4}\right)} \right) + 1$$

$$= \frac{4}{3} n^2 \left( 1 - \left(\frac{1}{4}\right)^k \right) + 1$$

$$\le \frac{4}{3} n^2 + 1 = O(n^2)$$

$$\therefore \quad T(n) = O(n^2).$$

Also

$$T(n) = \sum_{i=0}^{k-1} \left\lfloor \frac{n}{2^i} \right\rfloor^2 + 1$$

$$> \sum_{i=0}^{k-1} \left( \frac{n}{2^i} - 1 \right)^2 + 1 \quad \left\{ \begin{array}{l} \text{since} \\ \lfloor x \rfloor > x - 1 \end{array} \right.$$

$$= \sum_{i=0}^{k-1} \left( \frac{n^2}{4^i} - \frac{n}{2^{i-1}} + 1 \right) + 1$$

$$= n^2\left(\sum_{i=0}^{K-1}\left(\frac{1}{4}\right)^i\right) - 2n\left(\sum_{i=0}^{K-1}\left(\frac{1}{2}\right)^i\right) + K + 1$$

$$= n^2\left(\frac{1-\left(\frac{1}{4}\right)^K}{1-\frac{1}{4}}\right) - 2n\left(\frac{1-\left(\frac{1}{2}\right)^K}{1-\frac{1}{2}}\right) + K + 1$$

$$= \frac{4}{3}n^2\left(1-\frac{1}{4^{\lfloor \lg n \rfloor}}\right) - 4n\left(1-\frac{1}{2^{\lfloor \lg n \rfloor}}\right) + \lfloor \lg n \rfloor + 1$$

$$> \frac{4}{3}n^2\left(1-\frac{1}{4^{\lg n - 1}}\right) - 4n\left(1-\frac{1}{2^{\lg n}}\right) + (\lg n - \cancel{1}) + \cancel{1}$$

$$\left\{ \text{Since } \begin{array}{l} \lfloor x \rfloor > x-1 \\ \lfloor x \rfloor \le x \end{array} \right.$$

$$= \frac{4}{3}n^2\left(1-\frac{4}{n^2}\right) - 4n\left(1-\frac{1}{n}\right) + \lg n$$

$$= \frac{4}{3}n^2 - \frac{16}{3} - 4n + 4 + \lg n$$

$$= \frac{4}{3}n^2 - 4n + \lg n - \frac{4}{3}$$

$$= \Omega(n^2)$$

$$\therefore\ T(n) = \Omega(n^2)$$

$$\therefore\ T(n) = \Theta(n^2).$$

In General we deal with Recurrences of the form:

$$T(n) = \begin{cases} c & 0 \le n < n_0 \\ aT\left(\frac{n}{b}\right) + f(n) & n \ge n_0 \end{cases}$$

$\lfloor \frac{n}{b} \rfloor$ or $\lceil \frac{n}{b} \rceil$

## Master Theorem

Let $a \geq 1$, $b > 1$, $f(n)$ asymptotically positive and let $T(n)$ be defined by $T(n) = aT\left(\frac{n}{b}\right) + f(n)$.

Then we have 3 cases:

(1) If $f(n) = O\left(n^{\log_b(a) - \varepsilon}\right)$ for

some $\varepsilon > 0$, then

$$\boxed{T(n) = \Theta\left(n^{\log_b(a)}\right)}$$

(2) If $f(n) = \Theta\left(n^{\log_b a}\right)$ then

$$\boxed{T(n) = \Theta\left(n^{\log_b a} \cdot \log n\right) = \Theta\left(f(n) \cdot \log n\right)}$$

(3) If $f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right)$ for some $\varepsilon > 0$, and if

$$a\, f(n/b) \le c \cdot f(n)$$

for some $0 < c < 1$ and for all suff. large $n$, then

Regularity Condition

$$\boxed{T(n) = \Theta\left(f(n)\right)}.$$

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Rmks: In each case we compare

$$n^{\log_b a} \quad \text{to} \quad f(n)$$

The 'winner' determines Asymptotic Soln

here 'win' means win by Polynomial factor $n^{\varepsilon}$.

<u>note</u>: since Asymptotic soln only depends on $\Theta(f(n))$, we Sometimes write recurrance as

$$T(n) = aT\left(\frac{n}{b}\right) + \Theta(f(n))$$

<u>Ex.</u> $T(n) = 8T(n/2) + n^3$

Compare $n^{\log_2 8}$ to $n^3$

So by Case 2: $\boxed{T(n) = \Theta(n^3 \log n)}$

<u>Ex.</u> $T(n) = 5T(n/4) + n$

Compare: $n^{\log_4 5}$ to $n$.

let $\varepsilon = \log_4 5 - 1$. then since $5 > 4$, $\log_4 5 > 1$, so $\varepsilon > 0$. Also

$$n = n^1 = n^{\log_4 5 - \varepsilon} = O\left(n^{\log_4 5 - \varepsilon}\right)$$

By Case 1: $\boxed{T(n) = \Theta\left(n^{\log_4 5}\right)}$

<u>Ex.</u> $T(n) = 5T(n/4) + n^2$

Compare $n^{\log_4 5}$ to $n^2$

let $\varepsilon = 2 - \log_4 5$ . then since $5 < 16$, $\log_4 5 < 2$, so $\varepsilon > 0$.

Thus

$$n^2 = n^{\log_4 5 + \varepsilon} = \Omega\left(n^{\log_4 5 + \varepsilon}\right)$$

so we're in case 3. Must find c in range $0 < c < 1$ s.t.

$$5\left(n/4\right)^2 \leq c n^2$$

for suff. large n. i.e.

$$\frac{5}{16} \cancel{n^2} \leq c \cancel{n^2}$$

i.e. any $c$ in range

$$\frac{5}{16} \le c < 1$$

will work. So regularity cond. is satisfied. Hence

$$\boxed{T(n) = \Theta(n^2)}$$

Ex.

$$T(n) = 8T(n/2) + \underbrace{(10n^3 + 15n^2 - n^{1.5} + n\log n + 1)}_{= \Theta(n^3)}$$

then same as $1^{ST}$ example.

so $T(n) = \Theta(n^3 \log n)$

<u>note</u>: when $f(n)$ is a Polynomial M.T. is easy to apply. Just set

$$\varepsilon = \log_b a - \deg f \quad (\text{case } 1)$$

$$\varepsilon = \deg(f) - \log_b a \quad (\text{case } 3).$$

<u>Exercise</u>. Show that if $f(n)$ is a polynomial of deg $k$, and $k > \log_b a$ (so in Case 3), then regularity cond. holds.

Ex. ( f(n) not Polynomial )

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + \log(n!)$$

Simplify to

$$T(n) = 3T(n/2) + n \log n$$

Compare: $n^{\log_2 3}$ to $n \log n$.

Let $\varepsilon = \frac{1}{2}(\log_2 3 - 1)$. Then

Since $3 > 2$, $\log_2 3 > 1$, so $\varepsilon > 0$.

Also $1 + \varepsilon = \log_2 3 - \varepsilon$ so

$$\frac{n \log n}{n^{\log_2 3 - \varepsilon}} = \frac{n \log n}{n^{1+\varepsilon}} = \frac{\log n}{n^{\varepsilon}} \longrightarrow 0$$

so $n \log n = o\left(n^{\log_2 3 - \varepsilon}\right) \subseteq O\left(n^{\log_2 3 - \varepsilon}\right)$

so by case 1:

$$T(n) = \Theta\left(n^{\log_2 3}\right)$$

<u>Ex.</u> $T(n) = 2T(n/2) + \dfrac{n}{\log n}$

comp. $n^1$ to $\dfrac{n}{\log n}$