CMPS 201 4-14-10

Recall:

Thm: For All $n \geq 1$,
If T is a tree on n vertices,
Then T has $n-1$ EDGES.

Invalid Proof:

I. Same. ($n = 1 \Rightarrow T = \bullet$).

IIa. Let $n \geq 1$. let T be a tree on n vertices, AND ASSUME T has $n-1$ EDGES. ADD A new vertex to T AND Join it to any existing vertex in T with a new edge.

The Result is a Tree (still connected & acyclic) with $n+1$ vertices & $n = (n+1) - 1$ edges.

$\boxtimes$.

What's wrong with this?

$P(n)$ has form: $A(n) \rightarrow B(n)$ where

$A(n)$ is "T is a tree on n vertices"

$B(n)$ is "T has $n-1$ edges".

So IIa: $\forall n \geq 1 : P(n) \rightarrow P(n+1)$

Becomes $\forall n \geq 1$:

$$(A(n) \rightarrow B(n)) \rightarrow (A(n+1) \rightarrow B(n+1))$$

So we should:

let $n \geq 1$.

- Assume $A(n) \Rightarrow B(n)$;
- Assume $A(n+1)$
- show $B(n+1)$

we didn't do that. Instead we did following

- Assume T has n vertices & n-1 edges
- Construct a new tree from T with n+1 vertices & n edges.

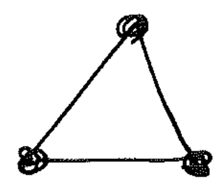
A false assertion proved same way:

- for all $n \geq 1$

false!!

If G is a conn. Graph on n vertices Then G has $n-1$ edges.

<u>NOTE</u>: obviously <u>false</u>!!



<u>Invalid Proof:</u>

I. let $n=1$. then G must be $\bullet$ which has 0 edges.

II. Let $n \geq 1$. let G be a conn. Graph on n vertices, and Assume G has $n-1$ edges.

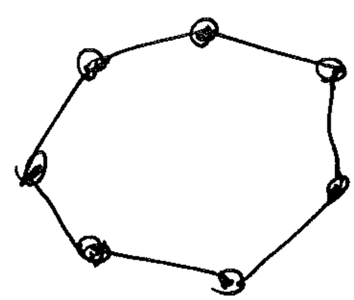
Add a new vertex and join it to any existing vertex in G by a new edge. The result is a new conn. Graph (Graphness & conn. are maintained) with n+1 vertices & n edges. $\boxtimes$

<u>Note</u>:

loop:

Disallowed by Defn of Graph.

cycle:



Allowed.

Recurrence Relations:

Ex $T(n) = \begin{cases} 0 & n = 1 \\ T(\lfloor n/2 \rfloor) + 1 & n \geq 2 \end{cases}$

Recall we showed $T(n) = O(\lg n)$.

lets solve recurrence by

Iteration:

$$\begin{aligned} T(n) &= 1 + T(\lfloor n/2 \rfloor) \\ &= 1 + \left(1 + T\left(\left\lfloor \frac{\lfloor n/2 \rfloor}{2} \right\rfloor\right)\right) \\ &= 1 + 1 + T(\lfloor n/2^2 \rfloor) \\ &= 1 + 1 + \left(1 + T\left(\left\lfloor \frac{\lfloor n/2^2 \rfloor}{2} \right\rfloor\right)\right) \\ &= 1 + 1 + 1 + T(\lfloor n/2^3 \rfloor) \end{aligned}$$

If necessary write

$$T(\text{☺}) = 1 + T\left(\left\lfloor \frac{\text{☺}}{2} \right\rfloor\right)$$

then replace ☺ by $\left\lfloor \frac{n}{2} \right\rfloor$.

then replace ☺ by $\left\lfloor n/2^2 \right\rfloor$

⋮

Continuing

$$T(n) = \underbrace{1 + 1 + \dots + 1}_{k} + T\left(\left\lfloor n/2^k \right\rfloor\right)$$

recursion depth

Process terminates when:

$$\lfloor n/2^K \rfloor = 1$$

i.e. $1 \le \frac{n}{2^K} < 2$

i.e. $2^K \le n < 2^{K+1}$

i.e. $K \le \lg(n) < K+1$

i.e. $K = \lfloor \lg(n) \rfloor$.

so for this K

$$T(n) = K + T(1)$$

i.e. $\boxed{T(n) = \lfloor \lg n \rfloor}$

Thus $T(n) = \Theta(\log n)$

Exercise:

Define $S(n)$ for $n \in \mathbb{Z}^+$ By

$$S(n) = \begin{cases} 0 & n = 1 \\ S\left(\left\lceil \frac{n}{2} \right\rceil\right) + 1 & n \geq 2 \end{cases}$$

show that soln to this recurrence is $S(n) = \lceil \lg n \rceil$ ← exact soln

So Also $S(n) = \Theta(\log n)$.

Asymptotic soln

In General the recurrence

$$T(n) = \begin{cases} c & 1 \le n < n_0 \\ T\left(\frac{n}{2}\right) + d & n \ge n_0 \end{cases}$$

where $\frac{n}{2}$ denotes either $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$ and c, d, n_0 are arbitrary, has Asymptotic solution:

$$T(n) = \Theta(\log n) .$$

<u>EXERCISE</u>

Prove the above assertion by the iteration method. Hint: show that the exact solution is:

<u>case: $\lfloor \cdot \rfloor$</u> $\quad T(n) = d \cdot \left(\lg \left\lfloor \frac{n}{n_0} \right\rfloor + 1 \right) + c$

<u>case: $\lceil \cdot \rceil$</u> $\quad T(n) = d \cdot \lg \left\lceil \frac{n}{n_0 - 1} \right\rceil + c$

Ex. let $c=3, d=5, n_0=10$ in ABOVE

i.e.

$$T(n) = \begin{cases} 3 & 1 \le n < 10 \\ T(\lfloor \frac{n}{2} \rfloor) + 5 & n \ge 10 \end{cases}$$

$$\begin{aligned} T(n) &= 5 + T(\lfloor \tfrac{n}{2} \rfloor) \\ &= 5 + \left(5 + T\left(\left\lfloor \frac{\lfloor n/2 \rfloor}{2} \right\rfloor\right)\right) \\ &= 5 + 5 + T(\lfloor n/2^2 \rfloor) \\ &\vdots \\ &= \underbrace{5 + 5 + \cdots + 5}_{k} + T(\lfloor n/2^k \rfloor) \\ &= 5k + T(\lfloor n/2^k \rfloor) \end{aligned}$$

must determine first (i.e. smallest) k st.

$$1 \le \lfloor n/2^k \rfloor < 10 .$$

i.e. find smallest k s.t.

$$1 \le \frac{n}{2^k} < 10$$

i.e. find least k s.t.

$$n < 10 \cdot 2^k$$

i.e. $$\frac{n}{10} < 2^k$$

i.e. $$\lg\left(\frac{n}{10}\right) < k$$

Since we seek least such k, we must have

$$k-1 \le \lg\left(\frac{n}{10}\right) < k$$

$$\therefore \quad k-1 = \left\lfloor \lg\left(\frac{n}{10}\right) \right\rfloor$$

$$\therefore \quad k = \left\lfloor \lg\left(\frac{n}{10}\right) \right\rfloor + 1$$

for this k, we have $1 \le \lfloor \frac{n}{2^k} \rfloor < 10$

hence $T(\lfloor \frac{n}{2^k} \rfloor) = 3$.

$\therefore$ we have

$$T(n) = 5k + T(\lfloor n/\cancel{8}2^k \rfloor)$$

i.e.

$$T(n) = 5\left(\lfloor \lg(\tfrac{n}{10}) \rfloor + 1\right) + 3$$

i.e. $\boxed{T(n) = 5\lfloor \lg(\tfrac{n}{10}) \rfloor + 8}$

when $n \ge 10$. hence

$$T(n) = \Theta(\log n).$$

Ex. $T(n) = \begin{cases} 1 & n = 1 \\ T\left(\lfloor \frac{n}{2} \rfloor\right) + n^2 & n \geq 2 \end{cases}$

$$T(n) = n^2 + T\left(\lfloor n/2 \rfloor\right)$$

$$= n^2 + \left(\lfloor \tfrac{n}{2} \rfloor^2 + T\left(\left\lfloor \frac{\lfloor n/2 \rfloor}{2} \right\rfloor\right)\right)$$

$$= n^2 + \lfloor \tfrac{n}{2} \rfloor^2 + T\left(\lfloor n/2^2 \rfloor\right)$$

$$= n^2 + \lfloor \tfrac{n}{2} \rfloor^2 + \left(\lfloor \tfrac{n}{2^2} \rfloor^2 + T\left(\left\lfloor \frac{\lfloor n/2^2 \rfloor}{2} \right\rfloor\right)\right)$$

$$= n^2 + \lfloor \tfrac{n}{2} \rfloor^2 + \lfloor \tfrac{n}{2^2} \rfloor^2 + T\left(\lfloor n/2^3 \rfloor\right)$$

$$= n^2 + \lfloor \tfrac{n}{2} \rfloor^2 + \lfloor \tfrac{n}{2^2} \rfloor^2 + \left(\lfloor \tfrac{n}{2^3} \rfloor^2 + T\left(\left\lfloor \frac{\lfloor n/2^3 \rfloor}{2} \right\rfloor\right)\right)$$

$$= n^2 + \lfloor \tfrac{n}{2} \rfloor^2 + \lfloor \tfrac{n}{2^2} \rfloor^2 + \lfloor \tfrac{n}{2^3} \rfloor^2 + T\left(\lfloor n/2^4 \rfloor\right)$$

$\vdots$ at recursion depth k !

$$T(n) = \sum_{i=0}^{k-1} \left\lfloor \frac{n}{2^i} \right\rfloor^2 + T\left(\left\lfloor n/2^k \right\rfloor\right)$$

must determine k s.t.

$$\left\lfloor n/2^k \right\rfloor = 1$$

this implies : $k = \lfloor \lg_2 n \rfloor$.

for this k we have

$$\boxed{T(n) = \sum_{i=0}^{k-1} \left\lfloor \frac{n}{2^i} \right\rfloor^2 + 1}$$

next time will show

$$T(n) = \Theta(n^2) .$$

imagine could erase $\lfloor\ \rfloor$.

have

$$\sum_{i=0}^{k-1} \frac{n^2}{4^i} = n^2\left(\sum_{i=0}^{k-1} \left(\frac{1}{4}\right)^i\right)$$

$$= n^2\left(\frac{1-(1/4)^k}{1-(1/4)}\right)$$

$$= \frac{4}{3}n^2\left(1 - \frac{1}{4^{\lfloor \lg n \rfloor}}\right)$$

imagine we erase this floor:

get

$$\frac{4}{3}n^2\left(1-\frac{1}{4^{\lg n}}\right) = \frac{4}{3}n^2\left(1-\frac{1}{n^{\lg 4}}\right)$$

$$= \frac{4}{3}n^2\left(1-\frac{1}{n^2}\right) = \underbrace{\frac{4}{3}n^2 - \frac{4}{3}}_{\Theta(n^2)}.$$