CMPS 201 4-12-10

**Ex** show for all $n \geq 1$ that

$$\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$$

Proof:
Let $P(n)$ be .

I. Base : show $P(1)$ is true.

i.e. show

$$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2\cdot 1+1)}{6}$$

i.e. $1^2 = \frac{1\cdot 2\cdot 3}{6}$

i.e $1 = 1$

This is true.

II a. Show $\forall n \geq 1: P(n) \Rightarrow P(n+1)$.

Let $n \geq 1$. Assume $P(n)$ is true.

i.e. Assume that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad (*)$$

we must show $P(n+1)$ follows,

i.e. must show

$$\sum_{i=1}^{n+1} i^2 = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$$

so

$$\sum_{i=1}^{n+1} i^2 = \left(\sum_{i=1}^{n} i^2\right) + (n+1)^2 \qquad (**)$$

$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \qquad \left\{ \begin{array}{l} \text{By the} \\ \text{Induction} \\ \text{Hypothesis} \end{array} \right.$$

$$= (n+1)\left[\frac{n(2n+1)+6(n+1)}{6}\right]$$

$$= \frac{(n+1)(2n^2+n+6n+6)}{6}$$

$$= \frac{(n+1)(2n^2+7n+6)}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}$$

$$= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$$

showing that $P(n+1)$ Holds.

$\therefore$ By The P.M.I

$P(n)$ Holds for all $n \geq 1$.

///.

Ex. Let $x \in \mathbb{R}$, and $x \neq 1$.

Show for all $n \geq 0$ that

$$\boxed{\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}}$$

Proof: Let $P(n)$.

I. BASE. Show $P(0)$ is true.

i.e. $$\sum_{i=0}^{0} x^i = \frac{x^{0+1}-1}{x-1}$$

i.e. $$x^0 = \frac{x-1}{x-1}$$

i.e. $$1 = 1$$, which is true.

II a. Show $\forall n \geq 0: P(n) \to P(n+1)$.

Let $n \geq 0$. Assume $P(n)$ is true.

i.e. we assume, for this $n$, that

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}$$

we must show $P(n+1)$ is true, i.e.

$$\sum_{i=0}^{n+1} x^i = \frac{x^{n+2} - 1}{x - 1}$$

So

$$\sum_{i=0}^{n+1} x^i = \left(\sum_{i=0}^{n} x^i\right) + x^{n+1}$$

$$= \frac{x^{n+1} - 1}{x - 1} + x^{n+1} \quad \Big\} \text{ By the ind. hyp.}$$

$$= \frac{x^{n+1} - 1 + x^{n+1}(x-1)}{x-1}$$

$$= \frac{\cancel{x^{n+1}} - 1 + x^{n+2} - \cancel{x^{n+1}}}{x-1}$$

$$= \frac{x^{n+2} - 1}{x-1}$$

$\therefore$ By The P.M.I, Result follows for all $n \geq 0$.

Alt. Version of Ind Step:

IIb. Show $\forall n > 0 : P(n-1) \Rightarrow P(n)$

Let $n > 0$. Assume $P(n-1)$ is true.

i.e. we assume for this $n$, that

$$\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1}.$$

we must show $P(n)$ is true:

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}.$$

So

$$\sum_{i=0}^{n} x^i = \left(\sum_{i=0}^{n-1} x^i\right) + x^n$$

$$= \frac{x^n - 1}{x - 1} + x^n \quad \Bigg\{ \text{By the Ind. Hyp.}$$

$$= \frac{x^n - 1 + x^n(x-1)}{x-1}$$

$$= \frac{\cancel{x^n} - 1 + x^{n+1} - \cancel{x^n}}{x-1}$$

$$= \frac{x^{n+1} - 1}{x-1}.$$

Result follows By P.M.I.

///

<u>Ex.</u> Define $T(n)$ By

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ T(\lfloor \frac{n}{2} \rfloor) + 1 & \text{if } n \geq 2. \end{cases}$$

Prove that for all $n \geq 1$:

$$\boxed{T(n) \leq \lg(n)}$$

Hence $T(n) = O(\log n)$.

Proof: Let $P(n)$ be above boxed stmt.

I. Base: show $P(1)$ is true

i.e. show $T(1) \le \lg(1)$, i.e.

$0 \le 0$. ✓

II. Ind. show $\forall n > 1$:

$$\underbrace{[P(1) \wedge P(2) \wedge \dots \wedge P(n-1)]}_{\forall k : 1 \le k < n : P(k)} \Rightarrow P(n)$$

strong induction hyp.

Let $n > 1$. Assume for this $n$, and all $k$ in the range

$1 \le k < n$ that $P(k)$ is true, i.e. for all such $k$ that

$$T(k) \le \lg(k).$$

we must show that $P(n)$ holds, i.e.

$$T(n) \le \lg(n),$$

so

$$T(n) = T(\lfloor \tfrac{n}{2} \rfloor) + 1$$

$$\le \lg(\lfloor \tfrac{n}{2} \rfloor) + 1 \quad \left\{ \begin{array}{l} \text{By the ind.} \\ \text{hyp, setting} \\ k = \lfloor \tfrac{n}{2} \rfloor \end{array} \right.$$

$$\le \lg\left(\tfrac{n}{2}\right) + 1 \quad \left\{ \begin{array}{l} \text{since } \lfloor x \rfloor \le x, \\ \text{hence } \lg \lfloor x \rfloor \le \lg x. \end{array} \right.$$

$$= \lg n - \cancel{\lg 2} + \cancel{1}$$

$$= \lg(n) \qquad \text{Result follows. ///}$$

Exercise:

Define $S(n)$ for $n \in \mathbb{Z}^+$ By

$$S(n) = \begin{cases} 0 & \text{if } n = 1 \\ S(\lceil \frac{n}{2} \rceil) + 1 & \text{if } n \geq 2 \end{cases}$$

show: $\forall n \geq 1$: $S(n) \geq \lg(n)$.

hint: use fact that $\lceil x \rceil \geq x$.

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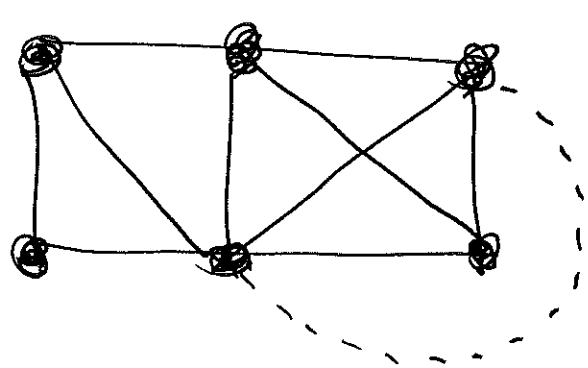
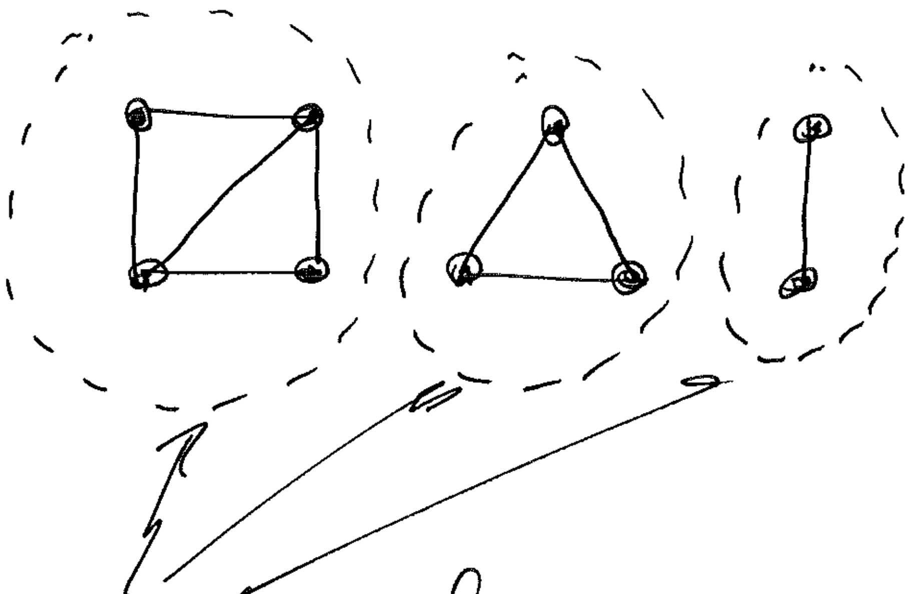
Examples of Graphs:

Connected Graph:



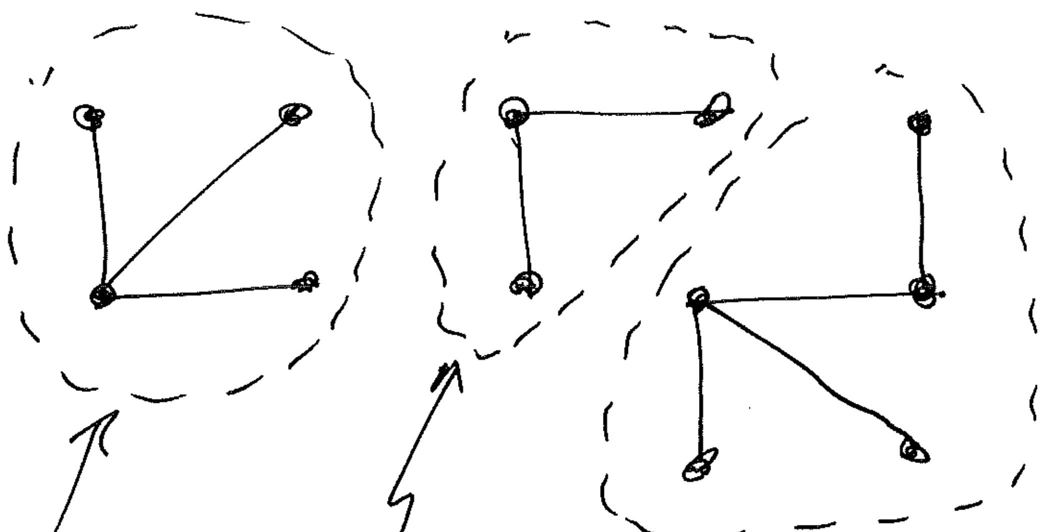
$|V| = 6$
$|E| = 9$

## Disconnected Graph :



connected Components.

## Acyclic Graph :



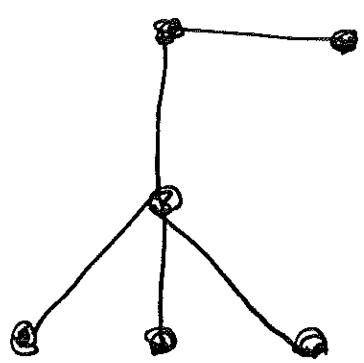
$n = 3$
$m = 2$

$n = 5$
$m = 4$

$n = 4$
$m = 3$

<u>Defn</u>: A <u>Tree</u> is a Graph which is both <u>acyclic</u> and connected.

<u>Ex</u>. Let $n = |V|$, $m = |E|$.



$n = 6$
$m = 5$

<u>Thm</u>. For all $n \geq 1$:

$$\boxed{\text{If } T \text{ is a tree on } n \text{ vertices, then } T \text{ has } n-1 \text{ edges}}$$

<u>Proof</u>: Induction on $n$ = #vertices

$P(n)$ is boxed stmt.

I. BASE. $P(1)$ says that all trees with 1 vertex, have 0 edges. there is only one tree (in fact only one graph) with 1 vertex:

Notice this tree has no edges.

II d. show $\forall\, n > 1$:

$$[P(1) \wedge P(2) \wedge \ldots \wedge P(n-1)] \rightarrow P(n)$$

i.e. $[\forall\, k : 1 \leq k < n : P(k)] \rightarrow P(n)$.

Let $n > 1$. Assume for all $k$ in the range $1 \le k < n$ that: If $T$ is a tree on $k$ vertices, then $T$ has $k-1$ edges. We must show: If $T$ is a tree on $n$ vertices, then $T$ has $n-1$ edges.

Assume $T$ is a Tree on $n$ vertices. Pick any edge in $T$ and Remove it. The result is Two subtrees $T_1, T_2$ each with fewer vertices than $T$. Say $T_i$ has $k_i$ vertices $(i=1,2)$.

Then $1 \le k_1 < n$ and $1 \le k_2 < n$. By the ind. Hyp:

$T_i$ has $k_i - 1$ edges.

$(i=1,2)$. Thus # edges in $T$ is

$$(k_1 - 1) + (k_2 - \cancel{1}) + \cancel{1} = k_1 + k_2 - 1$$
$$= n - 1,$$

As Required. ///.

## Problem 1 from HW 2



Show

$$\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$$

Suff to show that

$$L = \lim_{n \to \infty} \frac{\binom{2n}{n}}{\left(\frac{4^n}{\sqrt{n}}\right)}$$

exists, and $0 < L < \infty$