CMPS 201 3-31-10

Recall:

EXERCISE: let $g(n)$ be any a.n.n. fcn., and let $a > 0$. Then

$$a \cdot g(n) = O(g(n))$$ ← we'll show this now.

and $$a \cdot g(n) = \Omega(g(n))$$

so $$a \cdot g(n) = \Theta(g(n))$$

must show $\exists$ pos. c, n_0 s.t. $\forall n \geq n_0$:

$$0 \leq a \cdot g(n) \leq c \cdot g(n)$$

Pick $n_0 =$ and $c = a$.

whatever it has to be so that $g(n) \geq 0$ for all $n \geq n_0$.

Ex. show $\sqrt{n+10} = \Theta(\sqrt{n})$.

we must find Pos. c_1, c_2, n_0 s.t. for all $n \geq n_0$: $\boxed{0 \leq c_1\sqrt{n} \leq \sqrt{n+10} \leq c_2 \cdot \sqrt{n}}$.

Pick $c_1 = 1$, $c_2 = \sqrt{2}$, $n_0 = 10$.

Then for any $n \geq n_0$, we have:

$$-10 \leq 0 \quad \text{and} \quad 10 \leq n$$

$$\therefore \quad -10 \leq (1-1)n \quad \text{and} \quad 10 \leq (2-1)n$$

$$\therefore \quad -10 \leq (1-c_1^2)n \quad \text{and} \quad 10 \leq (c_2^2-1)n$$

$$\therefore \quad c_1^2 n \leq n+10 \quad \text{and} \quad n+10 \leq c_2^2 n$$

$$\therefore \quad 0 \leq c_1^2 n \leq n+10 \leq c_2^2 n$$

$$\therefore \quad 0 \leq c_1\sqrt{n} \leq \sqrt{n+10} \leq c_2\sqrt{n}.$$

///

check: $c_1 = \sqrt{\frac{1}{2}}$, $c_2 = \sqrt{\frac{3}{2}}$, $n_0 = 20$ Also works.

<u>EXERCISE</u>: Let $a, b \in \mathbb{R}$, $b > 0$.

Prove that $(n+a)^b = \Theta(n^b)$.

<u>Thm</u>. If <u>$h(n) = O(g(n))$</u> and if <u>$f(n) \le h(n)$</u>
<u>for all suff. large n</u>, then

$$f(n) = O(g(n)).$$

<u>PROOF</u>: our hypotheses say

① $\exists$ pos c_1, n_1 s.t. $\forall n \ge n_1$: $0 \le h(n) \le c_1 \cdot g(n)$

② $\exists$ pos n_2 s.t. $\forall n \ge n_2$: $0 \le f(n) \le h(n)$.

we must show $\exists$ pos c, n_0 s.t. $\forall n \ge n_0$:

$$0 \le f(n) \le c \cdot g(n)$$

Pick $n_0 = \max(n_1, n_2)$ and $c = c_1$. ///

Exercise

① If $h(n) = \Omega(g(n))$ and $f(n) \geq h(n)$ for suff. large n, then

$$f(n) = \Omega(g(n))$$

② If $h_1(n) = \Omega(g(n))$, $h_2(n) = O(g(n))$, AND $h_1(n) \leq f(n) \leq h_2(n)$ for all suff. large n, then

$$f(n) = \Theta(g(n))$$

Ex. Let $k \geq 1$ be a fixed integer. Show that

$$\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$$

Recall some formulas:

$$\sum_{i=1}^{n} i^1 = \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n \underset{\substack{\uparrow \\ \text{will Prove}}}{=} \Theta(n^2)$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \underset{\substack{\uparrow \\ \text{will Prove}}}{=} \Theta(n^3)$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \Theta(n^4)$$

$\vdots$

<u>Proof</u>: observe

$$\sum_{i=1}^{n} i^K \le \sum_{i=1}^{n} n^K = n \cdot n^K = n^{K+1} = O(n^{K+1})$$

Also

$$\sum_{i=1}^{n} i^K \ge \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^K$$

$$\ge \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \left\lceil \frac{n}{2} \right\rceil^K$$

$$= \left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right) \left\lceil \frac{n}{2} \right\rceil^K$$

$$= \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \cdot \left\lceil \frac{n}{2} \right\rceil^K \quad \left\{ \begin{array}{l} \text{since} \\ \lfloor n/2 \rfloor + \lceil n/2 \rceil = n \end{array} \right.$$

$$> \left(\left(\frac{n}{2} - \cancel{1}\right) + \cancel{1}\right) \cdot \left(\frac{n}{2}\right)^K \quad \left\{ \begin{array}{l} \text{since} \\ \lfloor x \rfloor > x - 1 \\ \lceil x \rceil \ge x \end{array} \right.$$

$$= \left(\frac{1}{2}\right)^{K+1} \cdot n^{K+1} = \Omega(n^{K+1}).$$

///

Notation:

- we write formulas like

$$h(n) = n^3 + 3n^{5/2} + \underbrace{\Theta(n^2)}$$

stands for some anonymous fcn in the class $\Theta(n^2)$.

- what is meant by:

$$\sum_{i=1}^{n} \Theta(i)$$

$$\underbrace{\Theta(1) + \Theta(2) + \Theta(3) + \dots + \Theta(n)}$$

This makes no sense

It means $\sum_{i=1}^{n} f(i)$ where

$f(i) = \Theta(i)$

<u>Exercise</u>: show that

$$\sum_{i=1}^{n} \Theta(i) = \Theta(n^2)$$

<u>Defn</u>

$$o(g(n)) = \{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) < c \cdot g(n) \}$$

<u>Recall</u>

$$O(g(n)) = \{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n) \}$$

Obviously: $o(g(n)) \subseteq O(g(n))$.

<u>Lemma</u>: $f(n) = o(g(n))$ iff

$$\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = 0$$

<u>Proof</u>: observe $f(n) = o(g(n))$ iff

$$\forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq \frac{f(n)}{g(n)} < c.$$

This is the very defn of limit stmt. ///

<u>Ex.</u> $\ln(n) = o(n)$ since

$$\lim_{n \to \infty} \left(\frac{\ln(n)}{n} \right) = \lim_{n \to \infty} \left(\frac{1/n}{1} \right) = 0.$$

Ex let $k > 0$. Then

$$n^k = o(e^n)$$

Since

$$\lim_{n\to\infty}\left(\frac{n^k}{e^n}\right) = \underbrace{\lim_{n\to\infty}\left(\frac{k(n^{k-1})}{e^n}\right) = \dots = 0}_{\text{AFTER } \lceil k \rceil \text{ APPLICATIONS of l'Hopitals rule.}}$$

Exercise:

$$o(g(n)) \cap \Omega(g(n)) = \emptyset$$

so

$$o(g(n)) \subseteq O(g(n)) - \Theta(g(n))$$

Picture so far