CMPS 201 3-29-10

Goal in analyzing time efficiency:

Determine $f(n)$ = # of basic operations performed on input of size n.

<u>Defn</u>: let $g(n)$ be a fcn. $O(g(n))$ is THE SET

$$\{ f(n) \mid \exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n) \}$$

i.e. $f(n) \in O(g(n))$ iff there are pos. c, n_0 s.t. for all $n \geq n_0$ we have

$$0 \leq f(n) \leq c\, g(n)$$

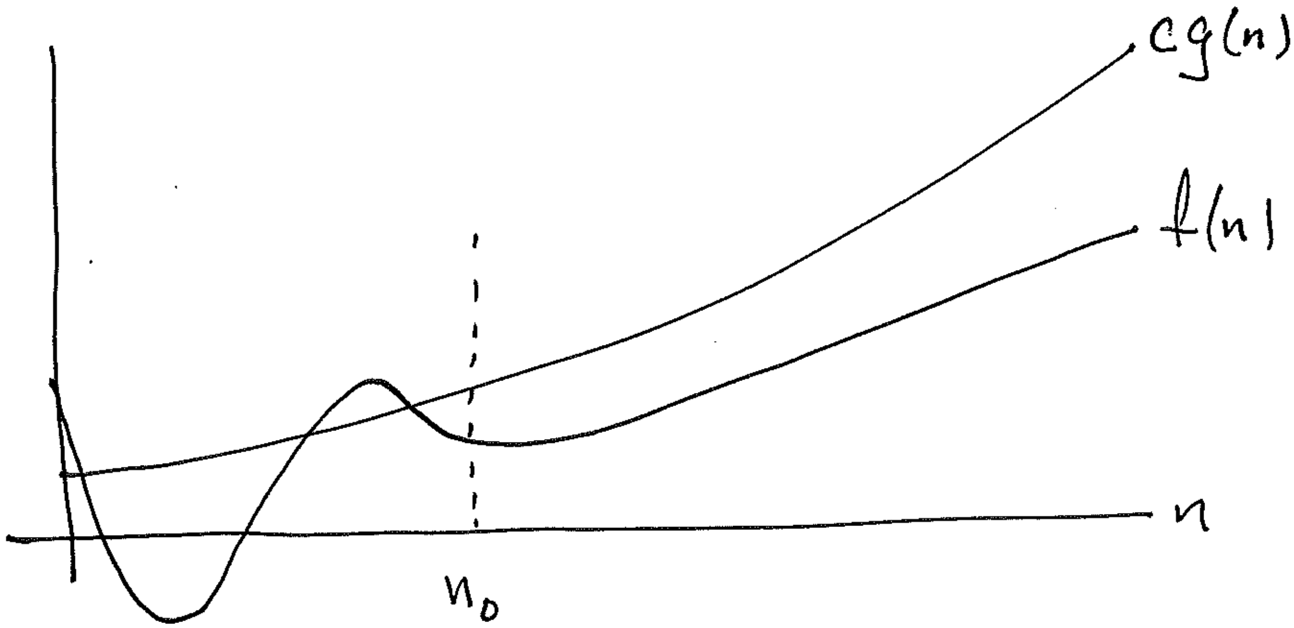
we say $g(n)$ is an <u>asymptotic upper bound</u> for $f(n)$

Abuse notation: write $\boxed{f(n) = O(g(n))}$ for $f(n) \in O(g(n))$.

really should write $f \in O(g)$.

In practice $f: \mathbb{Z}^+ \to \mathbb{Z}^+$. Its helpful to think of $f: \mathbb{R}^+ \to \mathbb{R}^+$.

Geometrically $f(n) = O(g(n))$ says:



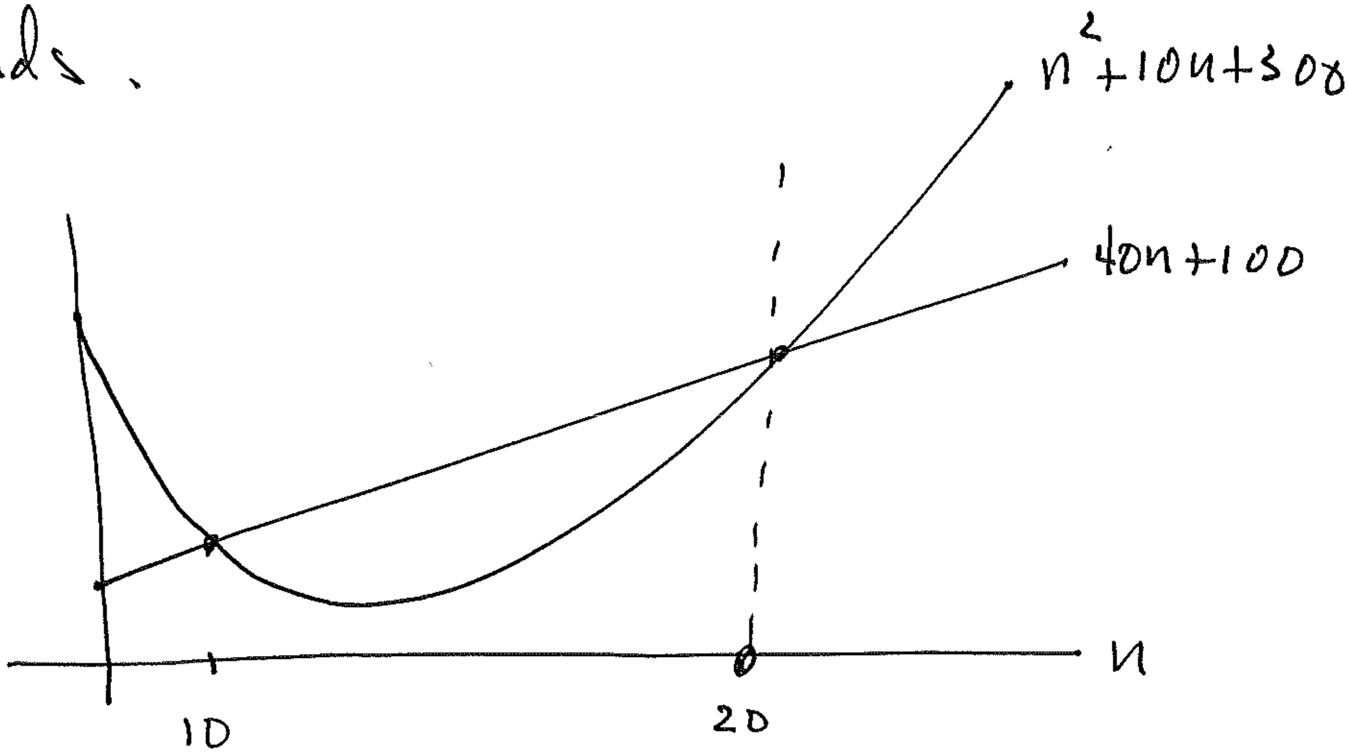
<u>Note</u>: if any c, n_0 work then infinitely many others also work.

<u>Ex.</u> $40n + 100 = O(n^2 + 10n + 300)$. why?

Must find pos c, n_0 s.t. for all $n \geq n_0$:

$$0 \leq 40n + 100 \leq c \cdot (n^2 + 10n + 300)$$

Holds.



Can take $c = 1$

$n_0 = 20$

<u>Defn</u>: $f(n)$ is <u>Asymptotically non-negative</u> if $\exists\, n_0 > 0$ s.t. $\forall n \geq n_0$: $f(n) \geq 0$.

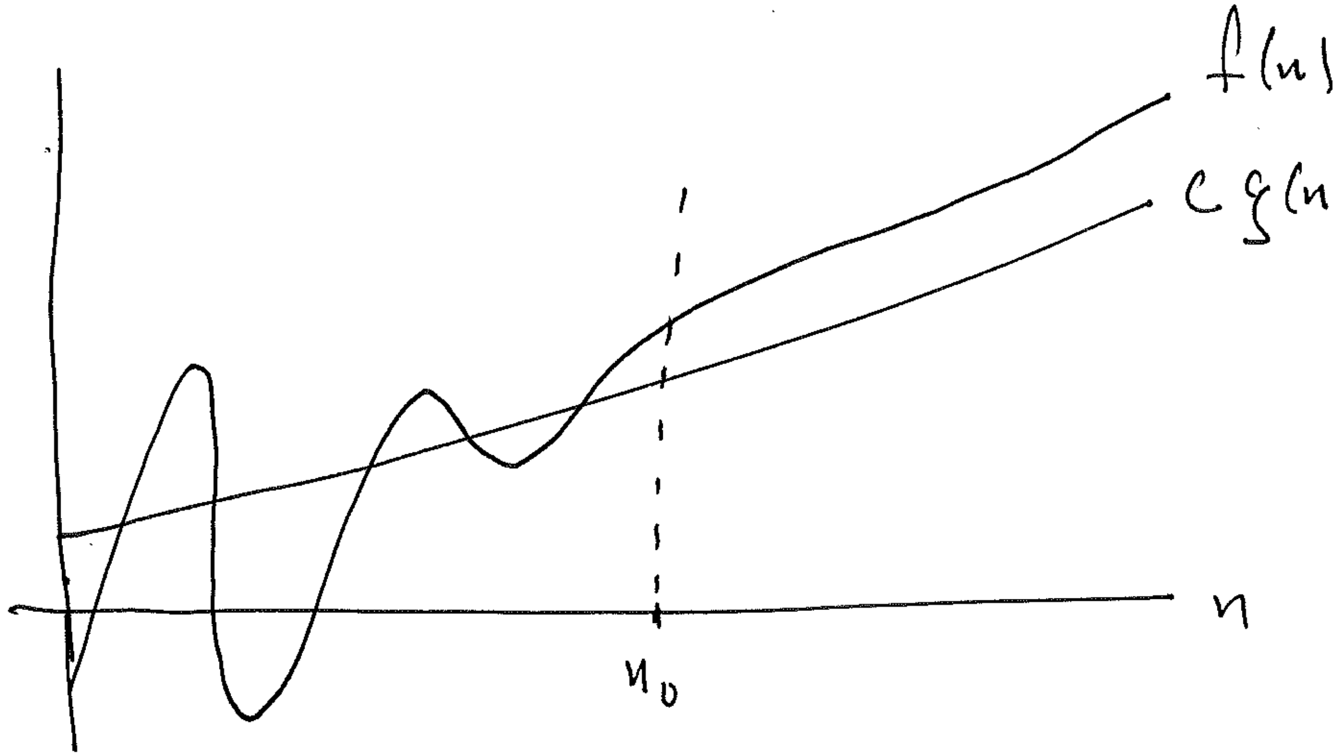
write: <u>a.n.n.</u>

<u>Defn</u> let $g(n)$ be a fcn. $\Omega(g(n))$ is THE SET

$$\{f(n) \mid \exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq c \cdot g(n) \leq f(n)\}$$

write: $f(n) = \Omega(g(n))$

Picture:



we say $g(n)$ is an <u>asymptotic lower bound</u> for $f(n)$.

Thm $f(n) = O(g(n))$ iff $g(n) = \Omega(f(n))$

Proof: ($\Rightarrow$) [($\Leftarrow$) is an exercise.]

If $f(n) = O(g(n))$ then there exist pos. numbers c_1, n_1 s.t. for all $n \geq n_1$

$$0 \leq f(n) \leq c_1 \cdot g(n)$$

Holds. We must specify pos. numbers c_2, n_2 s.t. for all $n \geq n_2$

$$0 \leq c_2 \cdot f(n) \leq g(n) . \qquad (*)$$

Let $c_2 = \frac{1}{c_1}$, and $n_2 = n_1$.

Then $c_2 > 0$ since $c_1 > 0$, and $n_2 = n_1 > 0$, and inequality $(*)$ holds for all $n \geq n_0$. ///

Analogy:

$f(n) = O(g(n)) \quad \sim \quad x \le y$

$f(n) = \Omega(g(n)) \quad \sim \quad x \ge y$

Defn let $g(n)$ be some (a.n.n.) fcn.

let $$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$$

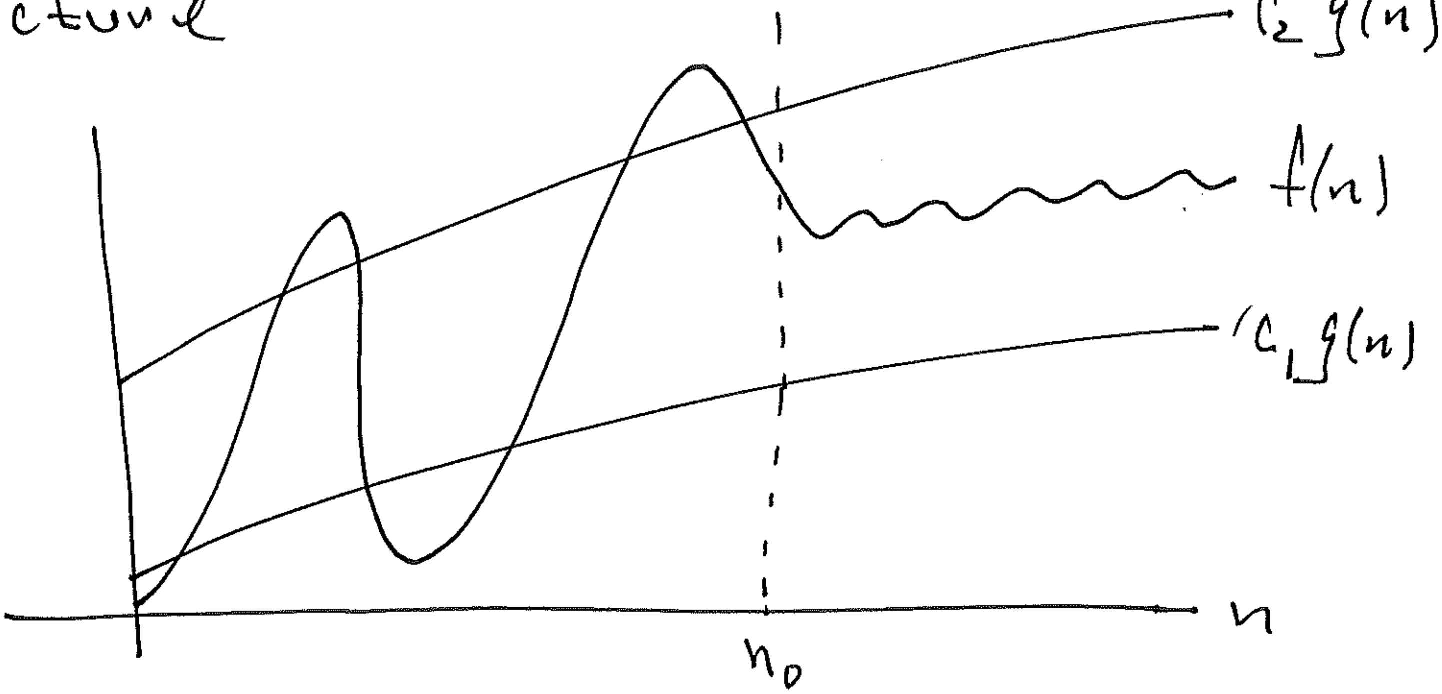
i.e. $\Theta(g(n))$ is the set

$$\{ f(n) \mid \exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0, \forall n \ge n_0 : \quad 0 \le c_1 \cdot g(n) \le f(n) \le c_2 \cdot g(n) \}$$

write: $f(n) = \Theta(g(n))$.

say: $g(n)$ is an asymptotically tight bound for $f(n)$

Picture



<u>EXERCISE</u> Prove that $f(n) = \Theta(g(n))$ iff $g(n) = \Theta(f(n))$.

<u>EXERCISE</u>: Let $g(n)$ be any (a.n.n.) fcn. and let $a > 0$. Prove that

$$a \cdot g(n) = \Theta(g(n)).$$