

### Problem Set 2

Due: Thursday, October 18, 2012

1. This problem works through the command optimum in continuous time. A planner seeks to maximize the utility function,

$$U_0 = \int_0^{\infty} e^{-\theta t} \frac{c^{1-\sigma} - 1}{1-\sigma} dt$$

with respect to its consumption plan,  $c_t$  for  $t \geq 0$ , subject to the resource identity,

$$\dot{k} = f(k) - \delta k - c,$$

given the initial capital-labor ratio  $k_0$ .

- Write out the formal constrained optimization problem for the planner.
- Use a current-time Hamiltonian to derive necessary conditions for an optimum. Be sure to write out all the necessary conditions.
- Draw a phase diagram to guide you through the problem. Show the path that  $c$  and  $k$  will follow in the solution. Find the steady state. Which necessary condition allowed you to select the path?
- Linearize the solution in a neighborhood of the steady state and confirm that the output growth rate is increasing in the difference  $y^* - y_0$  where  $y^*$  is steady-state output.

2. Consider an economy with endogenous capital and labor inputs in discrete time. The production function is  $y = f(k, n)$  and capital depreciates at the constant proportionate rate,  $\delta$ . The planner seeks to maximize the utility function,

$$U_0 = \sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t).$$

Labor supply,  $n_t$ , and leisure consumption,  $\ell_t$ , must each be nonnegative and satisfy  $0 \leq n_t + \ell_t \leq 1$ .

- Write out the constrained optimization problem for the planner.
- Write down the necessary conditions for an optimum. Now, let  $f(k, n) = Ak^\alpha n^{1-\alpha}$  for  $0 < \alpha < 1$  and  $A > 0$  constants. Let  $u(c, \ell) = \log c + \log \ell$ . Use these functions to rewrite your necessary conditions for this example economy.
- Using the conditions you derived in part b, find the equations that determine the steady-state variables. Write down a system of equations in  $c$ ,  $k$  and  $n$  for the dynamics of consumption, capital and labor. Linearize these around your steady state.
- Consider how a temporary increase in  $A$  (just for one period) affects each of these three variables when the economy is initially in the steady state. In particular, how do consumption and labor supply change in period  $t$  and period  $t + 1$  if  $A$  increases by  $\Delta A$  just for period  $t$ ?

3. The analysis of Tobin's  $q$  in the textbook is incomplete. In this problem, you will complete it.

a) Use the framework of section 2.7.1 and begin with equations 2.40 and 2.41. Rewrite these so that you have equations of motion for  $q$  and for  $k$  separately (so that  $\Delta k$  appears only in one equation and  $\Delta q$  only in the other).

b) Linearize the two equations about the steady state.

c) Show that the linearized system has a positive and a negative eigenvalue, and sign the slopes of the associated eigenvectors.

d) Let  $k_0$  be less than  $k^*$ , the steady-state capital stock. How do investment and  $q$  behave over time?

4. Reconsider the  $Ak$  model (constant returns to scale production with a single factor, capital) in discrete time. Let the utility function be given by

$$U_0 = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

for  $\sigma > 0$  and  $0 < \beta < 1$ . The resource identity is

$$\Delta k_{t+1} = Ak_t - \delta k_t - c_t.$$

a) Write the optimization problem using dynamic programming. Determine the necessary conditions for an optimum.

b) Use the necessary conditions to solve for the optimum. What is  $c_t$  as a function of  $k_t$ ?

c) Calculate utility at time  $t$ ,  $U_t = \sum_{s=t}^{\infty} \beta^{s-t} \frac{c_s^{1-\sigma} - 1}{1-\sigma}$ , in the optimum. Substitute your solution from part b to write the maximized solution for  $U_t$  as a function of  $k$ .

d) Verify that you have found the value function,  $V(k_t)$ .