Economics 205A
Oct 4, 2012

Some notes on the linear growth model

Start with the equation of motion for the capital stock,

$$k_{t+1} = (1-\delta)\, k_t + f(k_t) - c_t$$

and let $f(k_t) = Ak_t$ for a positive constant A. The equation of motion becomes

$$k_{t+1} = (1 + A - \delta)\, k_t - c_t$$

where we will denote $A - \delta$ by r, so that

$$k_{t+1} = (1+r)\, k_t - c_t.$$

The planner's objective is to maximize

$$U_0 = \sum_{t=0}^{\infty} \beta^t u(c_t).$$

We will let $u(c_t) = \log c_t$.

You can form the Lagrangian and derive the necessary conditions for an optimum,

$$\frac{1}{c_t} = \beta(1+r)\frac{1}{c_{t+1}},$$
$$k_{t+1} = (1+r)\, k_t - c_t$$

and

$$\lim_{T\to\infty} \beta^T \frac{1}{c_T} k_{T+1} = 0$$

given the initial capital stock equals k_0. We first rewrite the two equations of motion (the first-order conditions) as difference equations,

$$c_{t+1} - c_t = (\beta(1+r) - 1)\, c_t$$

and

$$k_{t+1} - k_t = rk_t - c_t,$$

and next write this dynamical system in matrix form as

$$\begin{bmatrix} \Delta c_{t+1} \\ \Delta k_{t+1} \end{bmatrix} = \begin{bmatrix} \beta(1+r) - 1 & 0 \\ -1 & r \end{bmatrix} \begin{bmatrix} c_t \\ k_t \end{bmatrix}.$$

You need to determine the eigenvalues of this system using the characteristic polynomial,

$$\det \begin{bmatrix} \beta(1+r) - 1 - \lambda & 0 \\ -1 & r - \lambda \end{bmatrix} = 0.$$

The eigenvalues are $\lambda_1 = \beta(1+r) - 1$ and $\lambda_2 = r$. You next find the eigenvector associated with each of the eigenvalues. These are

$$e_1 = \begin{bmatrix} (1-\beta)(1+r) \\ 1 \end{bmatrix} \quad \text{and} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

To find the optimum, you need to find a solution the difference equation system which satisfies the initial condition, $k = k_0$ at $t = 0$, and the transversality condition,

$$\lim_{T\to\infty} \beta^T \frac{1}{c_T} k_{T+1} = 0.$$

Since any solution to the linear dynamical system is a linear combination of the two solutions given by the eigenvectors,

$$\begin{bmatrix} c_t \\ k_t \end{bmatrix} = \begin{bmatrix} (1-\beta)(1+r) \\ 1 \end{bmatrix} (1+\lambda_1)^t k_0 \quad \text{and} \quad \begin{bmatrix} c_t \\ k_t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} (1+\lambda_2)^t k_0,$$

we first check which of these satisfies the transversality condition. Substituting in these solutions, the limits are

$$\lim_{T\to\infty} \beta^T \frac{1}{(1-\beta)(1+r)(1+\lambda_1)^T k_0} (1+\lambda_1)^{T+1} k_0 = \lim_{T\to\infty} \beta^T = 0$$

for the solution corresponding to $\lambda_1 = \beta(1+r) - 1$ and

$$\lim_{T\to\infty} \beta^T \frac{1}{0} (1+\lambda_2)^{T+1} k_0 = \lim_{T\to\infty} \beta^T \frac{1}{0} (1+r)^{T+1} k_0 \neq 0.$$

The optimum is given by

$$c_t = (1+\lambda_1)^t c_0 = (\beta(1+r))^t c_0$$

and

$$k_t = (1+\lambda_1)^t k_0 = (\beta(1+r))^t k_0$$

where

$$c_0 = (1-\beta)(1+r) k_0.$$

That is, $c_t = \theta\beta(1+r) k_t$ for all $t \geq 0$.

You should observe that if $\beta(1+r) < 1$, which means $\lambda_1 < 0$, then consumption and capital decrease over time at the per-period proportionate rate $|\lambda_1|$. In this case, consumption converges asymptotically to zero. If $\beta(1+r) > 1$ ($\lambda_1 > 0$), then consumption and capital increase over time at the per-period proportionate rate λ_1. In this case, consumption grows exponentially towards infinity with the capital stock. If $\beta(1+r) = 1$, then consumption is constant and equals $\left(\frac{1}{\beta} - 1\right) k_0$ and $k_t = k_0$ for all $t \geq 0$. For this special case, the two-dimensional dynamical system is degenerate,

$$\begin{bmatrix} \Delta c_{t+1} \\ \Delta k_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & r \end{bmatrix} \begin{bmatrix} c_t \\ k_t \end{bmatrix},$$

and the solution that satisfies the transversality condition is $\Delta k_{t+1} = \Delta c_{t+1} = 0$ so that $c_t = r k_t$ (where $r = \theta$).