

Midterm Exam Sample Answers

1. Use the discrete-time optimal growth model to answer this question. Assume production uses capital and labor under constant returns to scale, $Y = AF(K, N)$ and that representative household utility is

$$U_0 = \sum_{t=0}^{\infty} \beta^t \log c_t.$$

a) Write down the problem for finding the command optimum.

Ans: The command optimum is found by solving

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \log c_t$$

subject to

$$k_{t+1} = Af(k_t) + (1 - \delta)k_t - c_t,$$

$$k_{t+1} \geq 0 \quad \text{for all } t \geq 0$$

where k_0 is given. k_t denotes capital per capita.

b) Find the equations that determine the optimum.

Ans: The first-order conditions can be found by forming the Lagrangian,

$$L = \sum_{t=0}^{\infty} \beta^t (\log c_t + q_t (Af(k_t) + (1 - \delta)k_t - c_t - k_{t+1}))$$

and differentiating. These are the Euler condition,

$$c_t^{-1} = \beta (1 - \delta + Af'(k_{t+1})) c_{t+1}^{-1},$$

and the resource identity,

$$k_{t+1} = Af(k_t) + (1 - \delta)k_t - c_t.$$

The additional necessary conditions are the transversality condition, $\lim_{t \rightarrow \infty} \beta^t c_t^{-1} k_{t+1} = 0$ and the initial condition, k_0 given.

c) Suppose the economy is in steady state and productivity, A , rises temporarily (for just one period) by ΔA . How do consumption and investment respond? Explain how you would determine these responses analytically (show how if you have time).

Ans: We start by finding the steady state which is given by

$$Af'(k^*) - \delta = \beta^{-1} - 1 = \theta$$

and

$$c^* = Af(k^*) - \delta k^*.$$

A temporary rise in A at time t does not change the steady state and dynamics for the economy from $t + 1$

on. The rise in productivity increases resources at date t from $Af(k^*) + (1 - \delta)k^*$ to $(A + \Delta A)f(k^*) + (1 - \delta)k^*$. Define the equivalent level of k_t that would give us this output at time t under the production function, $Af(k_t)$, as

$$Af(\widehat{k}) + (1 - \delta)\widehat{k} = (A + \Delta A)f(k^*) + (1 - \delta)k^*.$$

All we need to do is find the point on the stable saddle path for initial capital stock, $k_t = \widehat{k}$, and we will have found c_t and the convergence path back to the steady state. The slope of the stable saddle path in a neighborhood of the steady state is the slope of the eigenvector, ν_- , associated with the negative eigenvalue, λ_- , of the linearized first-order conditions at the steady state. These are

$$\begin{bmatrix} c_{t+1} - c_t \\ k_{t+1} - k_t \end{bmatrix} = \begin{bmatrix} -\beta Af''(k^*)c^* & Af''(k^*)c^* \\ -1 & \theta \end{bmatrix} \begin{bmatrix} c_t - c^* \\ k_t - k^* \end{bmatrix}.$$

The slope of the stable saddle path is

$$\frac{c_t - c^*}{k_t - k^*} = \theta - \lambda_-.$$

To approximate the change in c_t , we can use the first-order approximation for $\widehat{k} - k^*$ which is

$$\widehat{k} - k^* = \frac{f(k^*)}{Af'(k^*) + (1 - \delta)} \Delta A = \beta f(k^*) \Delta A.$$

We have the change in c at time t ,

$$c_t - c^* = (\theta - \lambda_-) \beta f(k^*) \Delta A$$

which is positive (consumption rises at time t). The capital stock for $t + 1$ is given by substituting into the resource identity (you could also use the linearized Euler condition):

$$k_{t+1} + c_t = (A + \Delta A)f(k^*) + (1 - \delta)k^* \quad \Rightarrow \quad k_{t+1} - k^* = (1 - (\theta - \lambda_-)\beta) f(k^*) \Delta A = \frac{1 + \lambda_-}{1 + \theta} f(k^*) \Delta A.$$

Net investment at date t , $k_{t+1} - k^*$, is positive if $1 + \lambda_-$ is positive which it is if the optimum converges monotonically to the steady state.

2. Consider an economy in which output is produced using human capital and physical capital in continuous time. This output can be consumed or invested in physical capital. Human capital is produced using only human capital. The production function for output is given by

$$y_t = Ak_t^\alpha (\ell_t h_t)^{1-\alpha},$$

and the accumulation of human capital is given by

$$\dot{h}_t = B(1 - \ell_t)h_t - \delta h_t,$$

where ℓ_t is the allocation of time to producing goods. Let the representative household's utility be

$$U_t = \int_t^\infty \log c_s e^{-\theta(s-t)} ds.$$

a) Using this information, write down the complete problem for finding a command optimum.

Ans: The command optimum solves

$$\max_{\{c_s, k_s, h_s\}} \int_t^\infty \log c_s e^{-\theta(s-t)} ds$$

subject to

$$\begin{aligned} \dot{k} &= Ak_s^\alpha (\ell_s h_s)^{1-\alpha} - \delta k_s - c_s, \\ \dot{h} &= B(1 - \ell_s) h_s - \delta h_s, \\ 1 &\geq \ell_s \geq 0 \quad k_s \geq 0 \quad \text{and} \quad h_s \geq 0 \end{aligned}$$

for all $s \geq t$, given initial capital stocks, k_t and h_t .

b) Solve your problem for a balanced growth path. Define a balanced growth path. What is the balanced growth rate? What are the proportions of human to physical capital in the balanced growth path? How do the dynamics of consumption and capital of both kinds depend on the parameters in the problem?

Ans: The balanced growth found by first deriving the necessary conditions for the optimum. The current-time Hamiltonian is

$$H = \log c_s + q_1 \left(Ak_s^\alpha (\ell_s h_s)^{1-\alpha} - \delta k_s - c_s \right) + q_2 (B(1 - \ell_s) h_s - \delta h_s)$$

where we check that the inequality constraints on ℓ are satisfied later. An interior solution for ℓ satisfies the Euler conditions,

$$\begin{aligned} \dot{q}_1 &= \theta q_1 - q_1 \alpha A k^{\alpha-1} (\ell h)^{1-\alpha} + \delta q_1, \\ \dot{q}_2 &= \theta q_2 - q_1 (1 - \alpha) A k^\alpha (\ell h)^{-\alpha} \ell - q_2 B(1 - \ell) + \delta q_2, \end{aligned}$$

the first-order condition in the allocation of human capital, ℓ ,

$$\begin{aligned} q_1 (1 - \alpha) A k^\alpha (\ell h)^{-\alpha} h &= q_2 B h, \\ \frac{1}{c} &= q_1 \end{aligned}$$

the transversality conditions,

$$\lim_{t \rightarrow \infty} e^{-\theta t} q_{1t} k_t = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} e^{-\theta t} q_{2t} h_t = 0,$$

the two equations of motion,

$$\dot{k} = Ak^\alpha (\ell h)^{1-\alpha} - \delta k - c \quad \text{and} \quad \dot{h} = B(1 - \ell) h - \delta h$$

given the initial conditions.

A balanced growth path will have equal growth rates for k , h and c . Thus, $\frac{k}{h}$, $\frac{c}{k}$ and ℓ will be constant in the balanced growth path. The conditions above lead to

$$\frac{\dot{q}_2}{q_2} = \theta - B\ell - B(1 - \ell) + \delta = \theta - B + \delta$$

and

$$\frac{\dot{q}_1}{q_1} = \frac{\dot{q}_2}{q_2} \quad \Rightarrow \quad \alpha A k^{\alpha-1} (\ell h)^{1-\alpha} = B.$$

The growth rate is found using $\frac{1}{c} = q_1$ and $\frac{\dot{q}_1}{q_1} = \theta - B + \delta$, it is

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{\dot{h}}{h} = B - \delta - \theta.$$

We can find the ratio of k to h in the balanced growth path using the first-order condition for ℓ and the equation of motion for h ,

$$\alpha A k^{\alpha-1} (\ell h)^{1-\alpha} = B \quad \text{and} \quad \frac{\dot{h}}{h} = B(1-\ell) - \delta = B - \delta - \theta$$

which imply that

$$\ell = \ell^* \equiv \frac{\theta}{B} \quad \text{and} \quad \frac{k}{h} = \frac{k^*}{h^*} \equiv \frac{\theta}{B} \left(\frac{B}{\alpha A} \right)^{\frac{1}{\alpha-1}}.$$

The only condition we need for an interior solution is that $B \geq \theta$. We can also use the balanced growth rate and the equation of motion for physical capital to get the constant ratio of consumption to capital, $\frac{c}{k} = (1-\alpha) \frac{B}{\alpha} + \theta$.

The dynamics depend on the sign of $B - \delta - \theta$. If this growth rate is negative, the economy converges to the origin as k , h and c decrease exponentially toward zero. If it is positive, k , h and c increase forever exponentially. If it is zero, k , h and c remain at the initial values forever.

c) Assume that human capital cannot be transformed into physical capital and conversely. How is the optimal growth path determined when the initial stocks of capital, h_0 and k_0 , are not on the balanced growth path? Explain how the economy behaves if $\frac{h_0}{k_0}$ is less than the balanced growth ratio of h to k . You may find a diagram useful.

Ans: For this question, you need to explain that you would add the following constraints to the optimization problem:

$$\dot{h} \geq -\delta h \quad \text{and} \quad \dot{k} \geq -\delta k$$

but these just require $1 \geq \ell \geq 0$ and $c \geq 0$. Thus, $\frac{h_0}{k_0} > \frac{h^*}{k^*}$, $\ell_0 = 1$ and $\dot{h} = -\delta h$. If $\frac{h_0}{k_0} < \frac{h^*}{k^*}$, $\ell_0 = 0$ and $\dot{k} = -\delta k - c$. Beginning at time 0, the economy follows a transition path such that only investment in the short capital stock (for example, h if $\frac{h_0}{k_0} < \frac{h^*}{k^*}$) takes place until the ratio of human capital to physical capital reaches the balanced growth proportions $\frac{h^*}{k^*}$. When the ratio reaches the balanced growth proportions, the allocation of human capital, ℓ , jumps to its balanced growth level, ℓ^* . You might also write that the transition takes finite time because both h and k are positive in balanced growth (only convergence of k or h to zero would take infinite time).

A diagram would use the coordinates k and h and show the balanced growth path as an upward-sloping ray from the origin.

3. A representative household seeks to maximize its utility over consumption and leisure given by

$$U_t = \sum_{s=t}^{\infty} \beta^{s-t} \left(\log c_s + \frac{(1-n_s)^{1-\sigma}}{1-\sigma} \right)$$

where n_t denotes labor supply. The budget identity for the household is

$$a_{s+1} = (1 + r) a_s + w_s n_s - c_s,$$

where r is constant but w_t can vary with time.

a) Complete the household's optimization problem and find the necessary conditions for an optimum.

Use these to express c_s as a function of c_t and $(1 - n_s)$ as a function of $(1 - n_t)$.

Ans: The household optimum solves

$$\max_{\{c_s, n_s, a_{s+1}\}} \sum_{s=t}^{\infty} \beta^{s-t} \left(\log c_s + \frac{(1 - n_s)^{1-\sigma}}{1 - \sigma} \right)$$

subject to

$$a_{s+1} = (1 + r) a_s + w_s n_s - c_s,$$

$$\lim_{t \rightarrow \infty} \left(\frac{1}{1 + r} \right)^t a_t \geq 0$$

and

$$1 \geq n_s \geq 0 \quad , \quad c_s \geq 0,$$

for all $s \geq t$, given initial financial wealth a_0 . Assume $\sigma > 0$.

The necessary conditions for an optimum are

$$c_s^{-1} = \beta (1 + r) c_{s+1}^{-1},$$

$$(1 - n_s)^{-\sigma} = w_s c_s^{-1},$$

$$a_{s+1} = (1 + r) a_s + w_s n_s - c_s$$

and

$$\lim_{s \rightarrow \infty} \beta^{s-t} c_s^{-1} a_{s+1}$$

given initial financial wealth a_t .

To find c_s as a function of c_t , we solve the first-order condition for c backward to get

$$c_s = \beta^{s-t} (1 + r)^{s-t} c_t.$$

To find $1 - n_s$ we use the first-order conditions for c and $1 - n$ to get

$$\frac{(1 - n_s)^{-\sigma}}{w_s} = \beta (1 + r) \frac{(1 - n_{s+1})^{-\sigma}}{w_{s+1}}$$

and then solve backward to find

$$w_s (1 - n_s)^\sigma = \beta^{s-t} (1 + r)^{s-t} w_t (1 - n_t)^\sigma$$

which can be written as

$$(1 - n_s) = \left(\beta^{\frac{1}{\sigma}} (1 + r)^{\frac{1}{\sigma}} \right)^{s-t} \left(\frac{w_t}{w_s} \right)^{\frac{1}{\sigma}} (1 - n_t).$$

b) Find expressions for consumption and leisure at time t in terms of financial and human wealth.

Express these in terms of permanent income. (Hint: you can rewrite the budget identity by replacing $w_s n_s$

with $w_s - w_s(1 - n_s)$.)

Ans: Start by rearranging the budget identity as suggested and solving it forward and impose the transversality condition to get

$$\sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} [c_s + w_s(1 - n_s)] = (1+r)a_t + \sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} w_s.$$

In terms of permanent income, the budget constraint is

$$\sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} [c_s + w_s(1 - n_s)] = (1+r)a_t + \left(\frac{1+r}{r} \right) \tilde{w}_t \quad (3a)$$

where permanent income is

$$\tilde{w}_t = \frac{r}{1+r} \sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} w_s.$$

To find c_t , we need the goods and leisure consumption first-order condition,

$$c_s = w_s(1 - n_s)^\sigma,$$

to substitute c_s for $1 - n_s$ in expenditure, $[c_s + w_s(1 - n_s)]$, and the Euler condition for consumption,

$$c_s = \beta^{s-t} (1+r)^{s-t} c_t$$

to substitute for c_s in terms of c_t . Since this is an exam, you should stop here having explained how its done.

The full expressions for c_t is derived as

$$\sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} \beta^{s-t} (1+r)^{s-t} c_t + \sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} \left(\beta^{\frac{1}{\sigma}} (1+r)^{\frac{1}{\sigma}} \right)^{s-t} \left(\frac{w_t}{w_s} \right)^{\frac{1}{\sigma}} w_s(1 - n_t) = ra_t + \tilde{w}_t$$

which becomes

$$c_t \sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} \beta^{s-t} (1+r)^{s-t} + c_t^{\frac{1}{\sigma}} \sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} \left(\beta^{\frac{1}{\sigma}} (1+r)^{\frac{1}{\sigma}} \right)^{s-t} w_s^{\frac{\sigma-1}{\sigma}} = ra_t + \tilde{w}_t$$

then

$$c_t \frac{1}{1-\beta} + c_t^{\frac{1}{\sigma}} \sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} \left(\beta^{\frac{1}{\sigma}} (1+r)^{\frac{1}{\sigma}} \right)^{s-t} w_s^{\frac{\sigma-1}{\sigma}} = ra_t + \tilde{w}_t.$$

Using $c_t = w_t(1 - n_t)^\sigma$, the full expression for $1 - n_t$ is

$$w_t(1 - n_t)^\sigma \frac{1}{1-\beta} + w_t^{\frac{1}{\sigma}} (1 - n_t) \sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} \left(\beta^{\frac{1}{\sigma}} (1+r)^{\frac{1}{\sigma}} \right)^{s-t} w_s^{\frac{\sigma-1}{\sigma}} = ra_t + \tilde{w}_t.$$

These are simpler if we assume that $\beta = \frac{1}{1+r}$:

$$c_t + c_t^{\frac{1}{\sigma}} \left(\frac{r}{1+r} \right) \sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} w_s^{-\frac{1-\sigma}{\sigma}} = ra_t + \tilde{w}_t$$

and

$$w_t(1 - n_t)^\sigma + w_t(1 - n_t) \left(\frac{r}{1+r} \right) \sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} \left(\frac{w_t}{w_s} \right)^{\frac{1-\sigma}{\sigma}} = ra_t + \tilde{w}_t.$$

c) If a temporary productivity increase raises the wage rate at time t , how does this affect labor supply at time t ? If the household anticipates a permanent productivity increase beginning at time $t + 1$, how does this affect labor supply at time t ?

Ans: Start with what you think will happen. The rise in permanent income raises goods consumption at time t and in the future. For example, if $\beta = \frac{1}{1+r}$, an increase in permanent income increases consumption forever by the same amount. The rise in goods consumption leads to a rise in leisure consumption for all $s > t$ because $\frac{c_s}{(1-n_s)^\sigma} = w$. At time t , we use the same relationship, $\frac{c_t}{(1-n_t)^\sigma} = w_t = w + \Delta w_t$ to relate the change in goods consumption to the change in leisure consumption at time t . Letting the change in w equal an infinitesimal, dw_t , and differentiating, we get

$$\sigma \frac{dn_t}{(1-n_t)} = \frac{dw_t}{w} - \frac{dc_t}{c}.$$

We can see that if c_t does not rise by as much as w_t (proportionately), then we know that labor supply, n_t , rises at time t . Because the increase in w is temporary and consumption rises forever, we know consumption will not rise by as much as w_t . This is the permanent income model we worked out in Problem Set 3. You can use equation 3a

To do the algebra for this messy example, we start with equation 3a, substitute in the solved Euler condition for c from part b and simplify. Let's reduce the algebra by setting $\beta = \frac{1}{1+r}$ (this does not change the economics at all), and we get

$$c_t \left(\frac{1+r}{r} \right) + w_t (1-n_t) + \sum_{s=t+1}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} w (1-n_s) = (1+r) a_t + \frac{1}{r} w + w_t$$

where we used $1-\beta = \frac{r}{1+r}$, and use the first-order condition, $w(1-n_s)^\sigma = c_t$ (note, c_t is not a typo) for all $s > t$ and $w_t(1-n_t)^\sigma = c_t$, to get

$$c_t + \left(\frac{r}{1+r} \right) w_t \left(\frac{c_t}{w_t} \right)^{\frac{1}{\sigma}} + \left(\frac{1}{1+r} \right) w \left(\frac{c_t}{w} \right)^{\frac{1}{\sigma}} = r a_t + \frac{1}{1+r} w + \frac{r}{1+r} w_t.$$

Next, we differentiate with respect to w_t and c_t and evaluate at $w_t = w$:

$$dc_t + \left(\frac{r}{1+r} \right) \frac{\sigma-1}{\sigma} \left(\frac{c_t}{w} \right)^{\frac{1}{\sigma}} dw + \frac{1}{\sigma} \left(\frac{c_t}{w} \right)^{\frac{1-\sigma}{\sigma}} dc_t = \frac{r}{1+r} dw.$$

Rearranging once,

$$\left(1 + \frac{1}{\sigma} \left(\frac{c_t}{w} \right)^{\frac{1-\sigma}{\sigma}} \right) dc_t = \frac{r}{1+r} \left(1 + \frac{1-\sigma}{\sigma} \left(\frac{c_t}{w} \right)^{\frac{1}{\sigma}} \right) dw$$

and again,

$$\frac{dc_t}{c_t} = \frac{r}{1+r} \left(\frac{1 - \left(\frac{c_t}{w} \right)^{\frac{1}{\sigma}} + \frac{1}{\sigma} \left(\frac{c_t}{w} \right)^{\frac{1}{\sigma}}}{\frac{c_t}{w} + \frac{1}{\sigma} \left(\frac{c_t}{w} \right)^{\frac{1}{\sigma}}} \right) \frac{dw}{w}.$$

The only thing we might notice is that $1 - \left(\frac{c_t}{w} \right)^{\frac{1}{\sigma}} = \frac{c_t}{w} - \frac{r a_t}{w}$, so that the expression in brackets equals one if $a_t = 0$. (Solvency requires that $r a_t \leq w$, and we can show that $dc_t < dw_t$ for $r > 0$ and any a_t such that solvency holds.) Therefore, labor supply rises at time t and falls in time $t+1$ to its new permanent level. This household works more when the wage is temporarily higher and consumes more goods and leisure when the wage returns to its original level.

An anticipated wage increase of the same size at date $t+1$ will have just a slightly smaller effect on permanent income only because it is discounted by $\frac{1}{1+r}$ for one period. Consumption rises for all dates $s \geq t$

again and labor supply goes up with the wage for date $t + 1$. The only difference is that leisure will rise at time t along with c_t . That means that labor supply falls in time t and for all times $s \geq t + 2$.